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Tablet-based Technology to Support Students' Understanding of Division

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Abstract This paper reports on the design, implementation, and testing of a new tablet-based tool designed to help upper elementary students develop a strong understanding of division. The tool provides an interactive visual model for the process of division and leverages students' understanding of the array as a model for multiplication. Classroom observations, preliminary analysis of student work, and feedback from both students and teachers in a 4th grade classroom, indicate that the tool helped students increase their understanding of division and multiplication. The paper presents examples of student work that support this finding.

1 Problem Statement and Context

Many elementary school students struggle with division, and an inability to grasp and remember "long division" has been a tortuous part of many students' encounters with math in the upper elementary grades. This experience probably has persuaded many students that they are "bad at math." Division is difficult, but also important. According to a 2012 study, "Analyses of large, nationally representative, longitudinal data sets from the United States and the United Kingdom revealed that elementary school students' knowledge of fractions and of division uniquely predicts those students' knowledge of algebra and overall

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mathematics achievement in high school, 5 or 6 years later, even after statistically controlling for other types of mathematical knowledge, general intellectual ability, working memory, and family income and education" [8]. While this study doesn't prove a cause and effect relationship between understanding division and later achievement in mathematics, helping students master division likely provides them with important support and the confidence to navigate more complex mathematical topics later.

An understanding of multiplication is an important prerequisite for division. Division, however, is conceptually more difficult for students than multiplication. Multiplication involves three main quantities: the size of a group, the number of groups, and the total number obtained by multiplying the group size by the number of groups. Division too involves a total number of objects, a group size, and number of groups, but in the process of working through a division problem, a student must also keep track of the number of groups already accounted for, the number of objects the groups represent, and the number of objects still to be grouped, all of which change as more groups are formed. Understanding division is further complicated by the fact that the operation is used to solve two very different categories of problems [9]. Problems called *quotative* have a grouping structure. In these problems, the total number of objects in consideration and the size of the groups into which they are to be partitioned are known; the number of groups that can be created from the objects is unknown. An example of a quotative problem is: Kiri has 28 donuts and wants to put 7 on each plate. How many plates can she fill? Problems called *partitive* have a sharing structure. In these problems, the total number of objects in consideration is known, as is the number of groups, but the size of the groups is unknown. An example of a partitive problem is: Jonah has 28 donuts to divide among 7 friends. How many donuts will each friend receive?

While people who have mastered division are not necessarily aware of these two different problem structures, students first encountering quotative and partitive division problems may see the two types of problems as unrelated and may approach them quite differently. In working on quotative problems, which specify total number of objects and group size, students have a tendency to create multiple groups of the appropriate size iteratively, counting how many grouped objects they have and adding groups until they reach the total [2]. So, at least at the start, students see quotative problems as involving repeated addition and comparing the sum of the number of grouped objects with the total number of objects. The student work in Figure 1a, from a 4th grade class, illustrates this approach for a quotative problem.

Mastering partitive problems, which specify total number of objects and number of groups, requires students to understand that dealing out objects one at a time into groups results in an equal distribution of objects—a fact that is not obvious to many children. Once students realize this fact, they sometimes approach partitive problems by repeatedly subtracting one or more "rounds" of objects from the total until they have none left [2]. The student work in Figure 1b, from the same 4th grade class, illustrates this approach to a partitive problem.

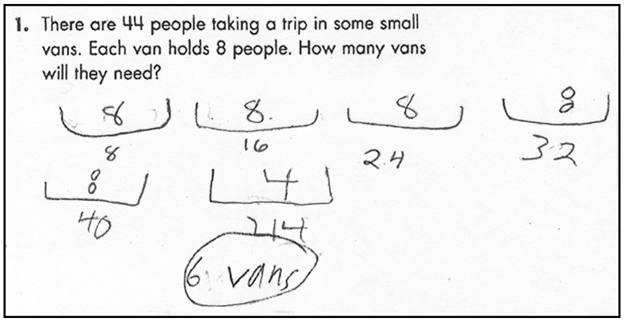

Fig. 1a Example of student work for a quotative problem (size of group given)

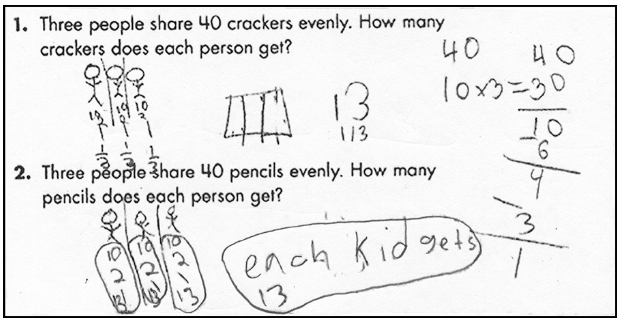

Fig. 1b Example of student work for a partitive problem (number of groups given)

These two examples of student work illustrate both the kinds of thinking students do when they work on division problems and the pitfalls of their strategies. While the examples demonstrate successful attempts at solving division problems, there are many examples of students who embark on the same path but lose track along the way, especially when a strategy requires a long string of steps. In one such example for 72 ÷ 4, shown in Figure 1c, a student successively drew groups of four, keeping track of the accumulating total after each new group. He added correctly until he reached 24, at which point he drew a group of four but added eight to his total by mistake.

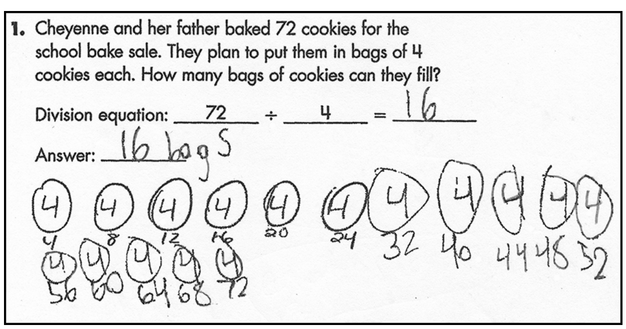

Fig. 1c Example of student losing track of counting in a division problem

Given both the difficulty and importance of division, helping students understand the conceptual basis of division is an important goal. Our work was guided by several perspectives on the learning of division: 1) Any approach to division, even the supposedly efficient long-division algorithm, has a bewildering set of numbers to keep track of. Whether the approach involves repeated addition or repeated subtraction, each time one more group is added or subtracted, several quantities change: the number of groups considered so far, the number of objects grouped so far, and the number of objects still to be grouped; 2) Understanding the relationship between multiplication and division is a crucial aspect of becoming proficient at division; and 3) Using arrays as a representation of multiplicative structures has the potential to help students see the relationship between multiplication and division and, eventually, to see the relationship between quotative and partitive division problems. This paper describes the tool we created to support students' understanding of division and our experiences using it with 4th grade students.

2 Related Work

Many of the educational software tools available for helping students learn division are "drill and practice" systems, which focus on developing mastery of multiplication and division facts rather than on understanding operations. Such systems, often with game-like formats, provide a series of structured exercises with immediate feedback and serve to review previously learned concepts rather than teaching new concepts. There are several tools, however, that focus on students' understanding of the operation of division, using visual models to reinforce the concept of grouping. One such example, Thinking Blocks (MathPlayground.com), supports students in creating bar models of division word problems. In these models, both dividend and divisor, i.e., group size, are represented as bars, and the task of dividing, at least for quotative problems, is figuring out how many divisor bars fit into the dividend bar. In the example shown in Figure 2, a student is asked to fill in a model and find a solution for a quotative problem: Given the number of straws to create a kite and a total number of straws, find the number of kites that can be made. A student is provided with a bar model template and blocks representing the concepts of total (straws), group size (straws per kite), and unknown number of groups (kites, represented by the "?"), as shown in Figure 2a. In a series of steps, the student adds both labels and numbers to the bar model, eventually arriving at Figure 2b, where she can type in her answer.

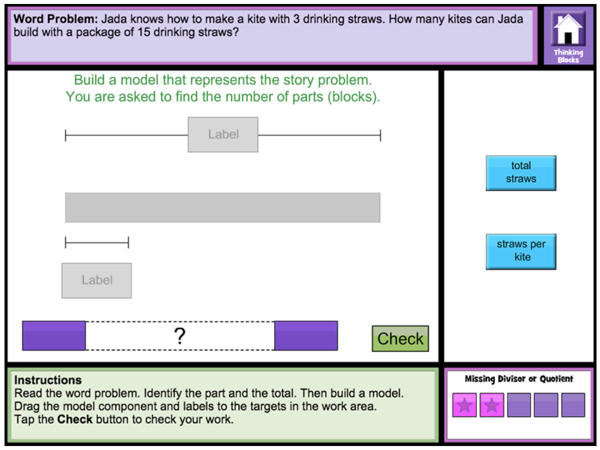

Fig. 2a Quotative problem with bar mode

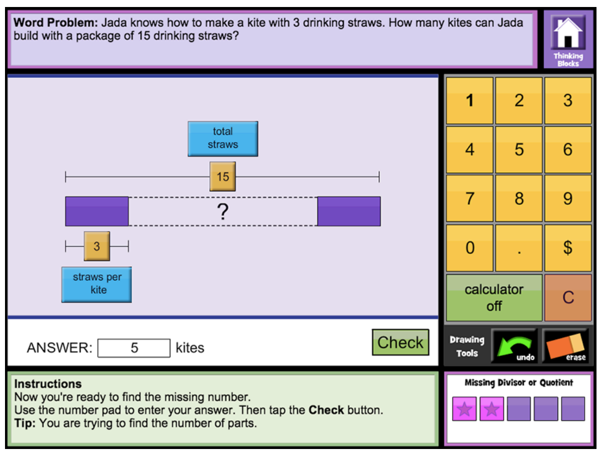

Fig. 2b Student positions bars and numbers, then enters answer via number pad

The Rectangle Division tool from the National Library of Virtual Manipulatives (nlvm.usu.edu) is another tool that uses a visual model and the concept of groups to support students' understanding of division. Instead of the one-dimensional bar model used in Thinking Blocks, this tool uses an array model in which the dividend is represented as an array on a grid. The student can set both dividend and divisor with arrow keys to the right and a slider to the left of the grid, respectively. Changing the dividend or divisor causes the quotient and remainder, represented as blue and red shaded array cells, respectively, to change to reflect the new values. (See Figure 3a.) Students also can use the tool as shown in Figure 3b, to request a division problem and create a visual representation to match it.

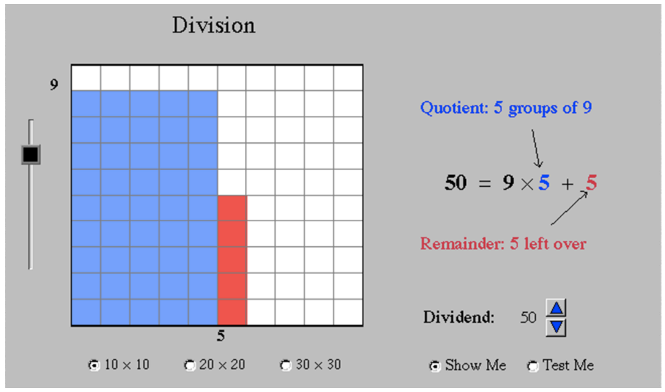

Fig. 3a 50 ÷ 9 modeled with arrays

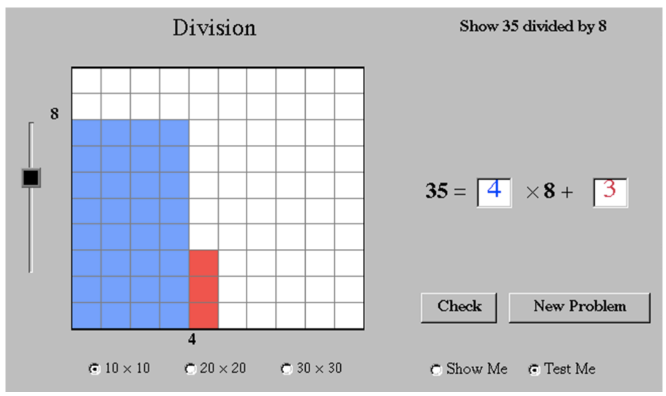

Fig. 3b Student is presented with a new problem

Dreambox (dreambox.com) also provides a tool that uses a visual grouping model for division. The Gumball Bag-O-Matic tool, shown in Figure 4a, specifies a problem in terms of a total number of gumballs and a bag of a specified size; the students are to find the number of bags needed. This tool prompts students to practice a strategy for computing an answer: doing smaller "friendly" sub-problems. The visual model represents the dividend as a group of gumballs and the divisor as a bag of a certain size. Students work on the problem by specifying a number of bags to fill using an equation template. The tool keeps track of the intermediate states of the problem by displaying to the left of a divider the accumulating sum of grouped gumballs, to the right of the divider the number of ungrouped gumballs, and above the model the number of bags created so far. When no more bags can be added, the quotient is the sum of the numbers at the top, and the remainder is the number to the right of the divider. (See Figure 4b.)

Fig. 4a Student is presented with 6762 ÷ 50

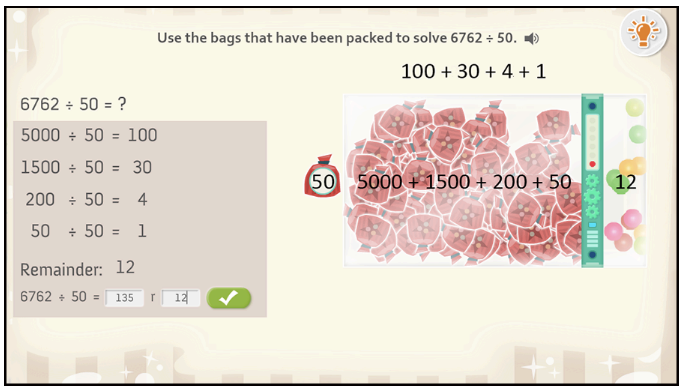

Fig. 4b Quotient is sum above model, remainder is on right

As described in the next section, our division tool shares features with each of these three examples, but also has some significant differences. Like the tools described above, our tool enables students to create visual models that reinforce the idea that division can be considered a process of figuring out the number of groups of a particular size that can be formed from a total. Our tool differs, however, in its consistent use of arrays for both dividend and intermediate sub-problems, as arrays explicitly support students in relating division to multiplication. Thus with our tool, students create sub-problems with arrays rather than with equations. Our tool enables students to create and modify their own models, resulting in a variety of student work that can serve as the basis for class discussion about alternate problem-solving strategies. Finally, because the tool is part of a tablet-based software system, students can annotate their models easily with a tablet pen in order to clarify and explain their work.

3 Method Employed

The new tool reported in this paper was designed to help students develop a strong understanding of division. The tool enables students to visualize the repeated addition of quantities to reach a total and helps students keep track of the computation during this process. It is part of a tablet-based software system we have been developing called Classroom Learning Partner (CLP) [3, 4, 5, 6]. CLP allows students to use a tablet pen to create answers to in-class problems and wirelessly send their answers to the teacher. Using CLP and a design-based research approach [1], we prototyped the new division tool, tested the tool in interviews with individual students and in a classroom setting, and used the results of testing to inform redesign.

3.1 CLP Background

In a classroom using CLP, all students and the teacher have tablets,[1] and a tablet is connected to a projector. All tablets communicate via a local wireless network. CLP can be thought of as an electronic notebook, with pages on which students show their work. In the current CLP classrooms, that work involves the creation and modification of mathematical representations used in upper elementary school. CLP provides tools to facilitate these operations, and the tablet pen plays an important role: The pen enables students to use a familiar mode of interaction, namely writing, while also enabling students to quickly and easily create and annotate structured mathematical objects such as arrays, number lines, and the division tool reported in this paper. (See [7] for discussion of the importance of drawing and annotation in young students' learning of multiplication and division.)

The CLP tool most closely related to the division tool is the array tool, which is used to represent multiplicative relationships. Tapping on an array tool icon on CLP's command bar, a user is prompted to input numbers for rows and columns using a displayed number pad, shown in Figure 5a.[2] An array of the specified size appears on the page. Shown in Figure 5b, is a 9x7 array that a student has created and annotated.

[1] Students use Lenovo X201s; the teacher uses a Microsoft Surface Pro. CLP is written in C# and runs on Windows.

[2] Handwriting recognition software proved too inaccurate to support entry via handwritten numbers.

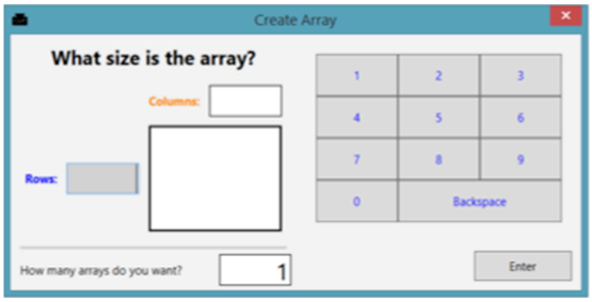

Fig. 5a UI for specifying number of rows and columns

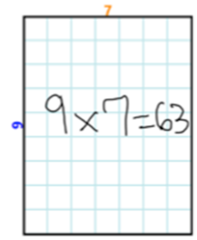

Fig. 5b 9x7 array created and annotated by a student

3.2 Division Template

CLP's division tool provides an interactive visual model for the division process and leverages students' understanding of the array as a model for multiplication. A division problem is represented by what we call a *division template*, which can be thought of as an array with one unspecified dimension, which represents the quotient. The unspecified dimension is indicated by a wavy line at the right edge of the division template.[3] The specified dimension represents the divisor; the area enclosed by the division template represents the dividend. The template is particularly well-suited for quotative problems. For such problems, arrays represent groups of the specified size, and the area of an array represents the number of grouped objects. So a 9x1 array represents one group of nine objects, a 9x2 array represents two groups of nine objects, and so on. To solve a division problem using a template, a student creates a template by specifying the dividend and divisor, which are termed "product" and "factor" in order to make explicit connection to students' understanding of multiplication and arrays. The student then overlays arrays on the template until the template is "full", i.e., until the sum of the arrays' areas reaches the dividend or the unfilled area of the template is less than the divisor. When a template is full, the divisor is represented by the number of rows, the quotient is represented by the number of columns, and any remainder is represented by the unfilled area in the template.

To give visual feedback and to facilitate moving the template on the screen, arrays are "snapped" into the template so that they become a part of the template itself. A student may create and snap in a variety of small arrays or a single large array. To help students keep track of the changing state of the division problem, the template shows the original dividend, the snapped-in arrays' area so far, and the remaining area needed to reach the dividend. It also gives feedback by displaying the template boundary in red if a student attempts to snap in an array that is too large or has the wrong dimensions.

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[3] Interviewed students thought that the wavy line was the clearest representation. Other designs included omitting the right edge entirely, dotting the right edge, or shading the edge so as to appear to fade into the background.

Figure 6 illustrates the use of a division template for a problem such as: Given 55 cookies and bags that hold 7 cookies, how many bags are needed? In this problem, the dividend of 55 cookies is represented as the total area of the template, the divisor of 7 cookies per bag is represented both as the number of rows in the template and as a 7x1 array, which represents the bag, and the quotient, i.e., the unknown number of bags, is represented by the number of columns needed to fill the division template. In Figure 6a, the student has created a division template representing the problem. The template shows the dividend of 55 in the center of the template and the divisor of 7 along the left side of the template. Figure 6b shows the result of snapping in a 7x1 array and a 7x2 array: The arrays are inside the template, the accumulated array area is reported below the template as the equation 7 x 3 = 21, and the remaining area of 34 is reported in the "empty" space of the template. Figure 6c shows the full division template, with four snapped-in arrays—one 7x1 and three 7x2 arrays. The division template reports the original dividend of 55, an array area total shown at the bottom as the equation 7 x 7 = 49, and a remaining area of 6. From this representation the student can identify that there are 7 groups of 7 in 55, with 6 left over, so 55 ÷ 7 is 7 remainder 6.

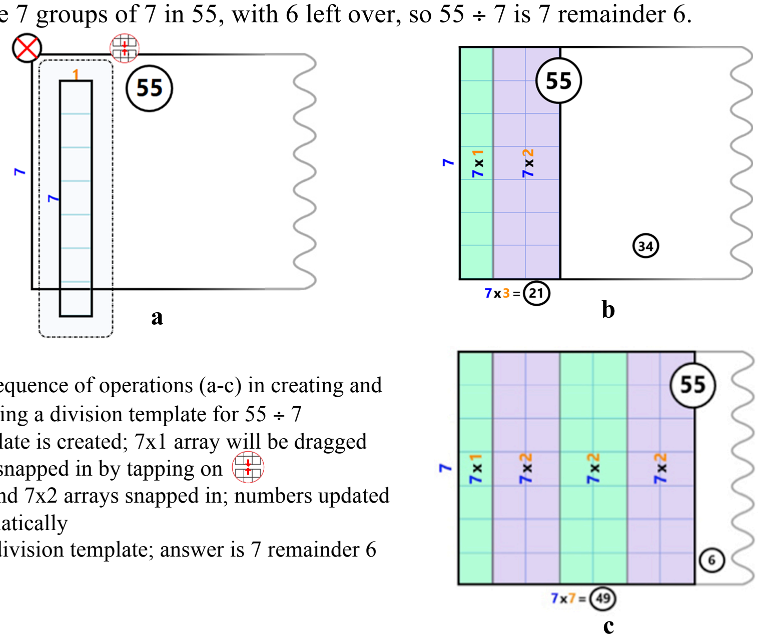
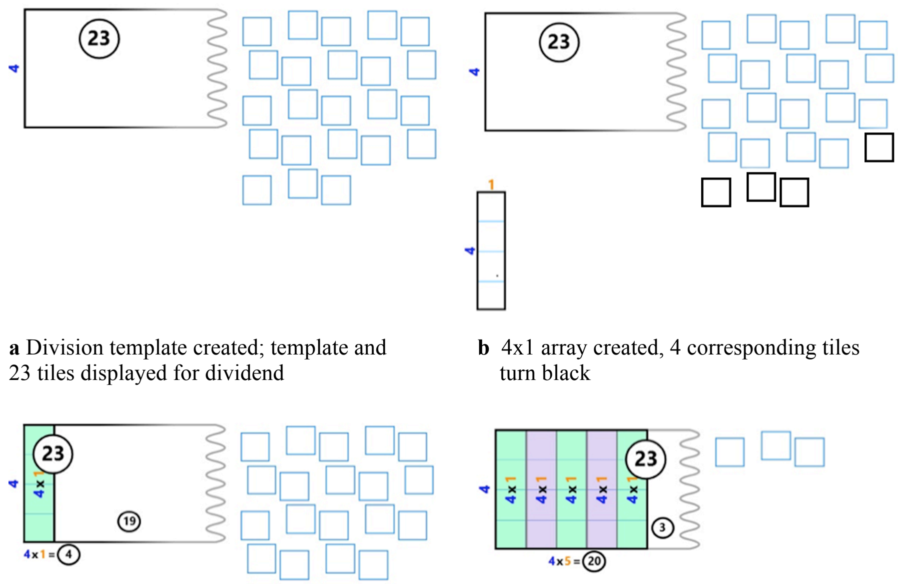


Fig. 6 Sequence of operations (a-c) in creating and filling a division template for 55 ÷ 7
a Template is created; 7x1 array will be dragged then snapped in by tapping on
b 7x1 and 7x2 arrays snapped in; numbers updated automatically
c Full division template; answer is 7 remainder 6

The above example is a quotative division problem. The template also can be used for partitive problems: The group size is represented by the number of columns and the number of groups is represented by the number of rows. Adding an array corresponds to adding objects to each of the groups.

In the course of developing the division tool, we realized that the concept of division could be made even clearer if students had a visual representation for the objects still to be grouped. We developed a second version of the tool that employs "tiles"—small squares the size of an array's cells. The use of this version of the tool is shown in Figure 7. The key idea is that the tiles reflect the current state of the division template: When the template is created, tiles are displayed as an additional representation for the dividend; as an array is created, the tiles corresponding to the array's area are automatically highlighted; then when an array is snapped into the template, the tiles automatically disappear as they are "added to" the template via the array.



a Division template created; template and 23 tiles displayed for dividend

b 4x1 array created, 4 corresponding tiles turn black

c 4x1 snapped in, corresponding tiles disappear

d Full template, tiles represent remainder

Fig. 7 Sequence of operations (a-d) in using a division template with tiles for 23 ÷ 4

3.3 Design Decisions

One of the early design decisions was to have the division template support students by keeping track of the quantities involved in division, especially those that change as the student works through the problem. Three such quantities are always connected by an additive relationship: the total (dividend), the number already added to the template, and the number still left. The second and third numbers always add up to the first. The division template emphasizes the quantities, via circling, in order to help students notice and track the relationship between the quantities. The division template also helps students keep track of the arrays they have added: Alternating colors for the arrays enable the individual arrays to be more easily seen, which makes it more obvious to the student that the array areas are being added to reach the total. The division template also uses color to distinguish the horizontal and vertical dimensions of the array: Anything referring to the vertical dimension is in blue, and anything referring to the horizontal dimension is in orange. This scheme means that all the blue numbers are the same—since all arrays that are snapped into the template have the same vertical dimension, i.e., number of rows. It also means that the orange numbers displayed inside the arrays always sum to the orange number shown in the equation below the template.

With any digital tool that supports students in developing computational fluency, there is always the question of how much computation to do for the student and how much to leave to the student. Our division template takes on much of the computational burden for the student, always updating the "number already added" and the "number still left." Doing this kind of bookkeeping for students runs the risk of making them dependent on the tool, but we decided that it was important for students to have this support in order for them to develop an overall model of division and confidence in their ability to navigate the process. A next step would be to gradually take away some of the support and transfer to the student more responsibility for keeping track of the computation.

We based the design of the division template on an array model for both multiplication and division for several reasons. Because students had used an array model for multiplication, carrying over this visual representation into the division template helped students to see the relationship between multiplication and division explicitly and to realize that knowledge of multiplication was relevant to division. Using arrays also sets the stage for students to eventually make the connection between quotative and partitive division contexts, as they can see that an array representing N groups of M is really the same structure as one representing M groups of N, only rotated [2, p. 54]. While we did not take advantage of this connection in our classroom work, we wanted to provide students with a model that would be able to support this understanding.

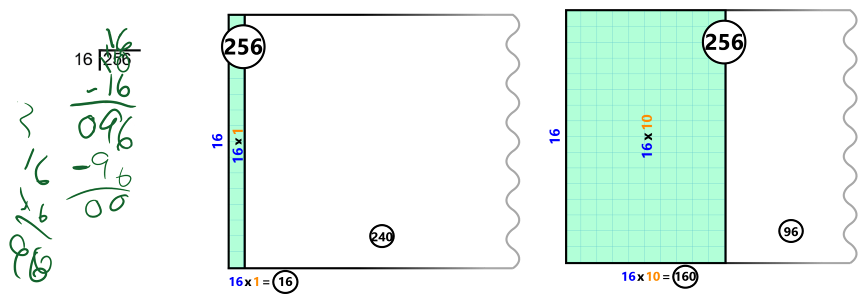
4 Results and Evaluation

We spent five weeks in a 4th grade classroom in Cambridge, MA, working with 19 students on a multiplication and division unit from TERC's *Investigations* curriculum, which we had adapted for use with our tablet-based software. Students did all of their math work on the tablets during this unit, allowing us to easily collect it, including re-playable interaction histories of their work on each problem. We took detailed field notes during class and interviewed several students at the end of the unit. During the last two weeks, the students used the division tool. Our ongoing analysis is qualitative in nature; our goal is to identify interesting patterns in the ways in which students use the division template in order to discover how the template may or may not support students' understanding of division.

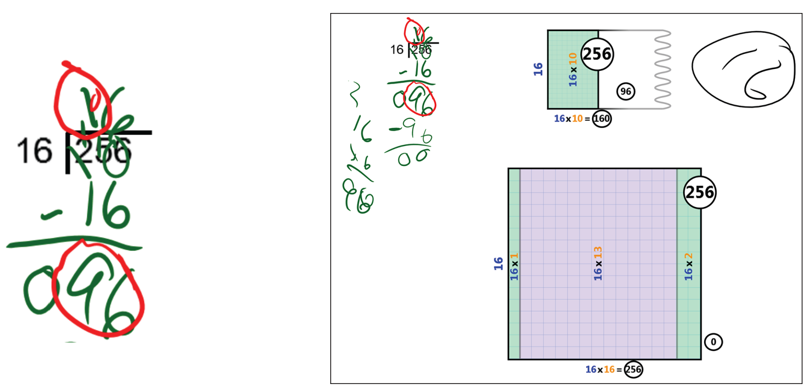
Classroom observations, preliminary analysis of student work, and feedback from both students and teachers, indicate that the tool helped students, particularly those struggling with the concept of remainder, increase their understanding of division and multiplication. The visual representations enabled some students to grasp concepts that were difficult for them when they had only algorithmic methods to solve problems. The tool provided a specific starting point for students unsure of how to approach a problem, enabled students to create representations quickly without perseverating over details, and enabled struggling students to work more confidently and independently. The tool also enabled students to make visible the variety of strategies that could be used to solve problems, which supported valuable class discussions of alternate problem-solving strategies. Below are examples that illustrate these findings.

Understanding Division

The division tool's visual representation helped some students understand a difficult concept that was obscured when just using an algorithm. Seth was quite fluent in his ability to do long division using the traditional algorithm. Shown in Figure 8a is his solution to 256 ÷ 16 via the traditional method. When we asked him in an interview also to do the problem using the division tool, he began with a 16x1 array, mimicking his "1" as the first digit of the solution (Figure 8b). When he saw what happened, he was puzzled, because, as he said, "I thought this [referring to his long division problem] would be the same as the division tool….but it's not." The interviewer proceeded to show the student what would happen if he had added a 16x10 array (Figure 8c) rather than a 16x1 array. For Seth, this step resulted in a major breakthrough in his understanding of place value in division, as he said, "I never thought of this [using red ink to circle the 1 and to add a 0 in his long division solution (Figure 8d)]—I just thought of this as a 1, but it's really a 10." The final state of his work page is shown in Figure 8e.

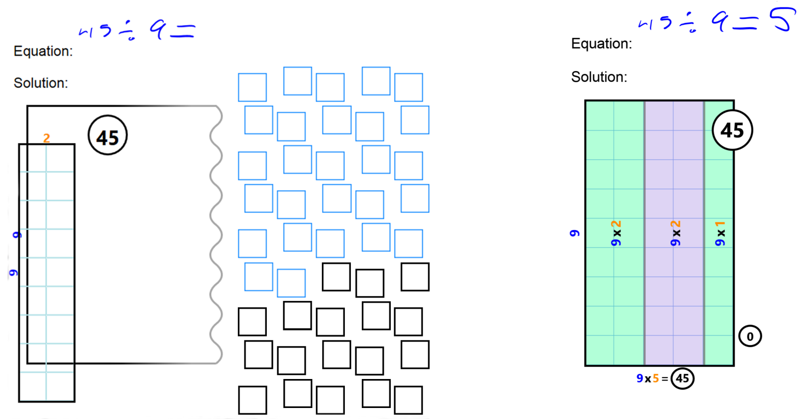


a Student's algorithm use **b** Student's partially filled template **c** Interviewer's suggestion



d Student's annotation **e** Final state of work page (template at top was resized)

The division template with tiles helped students better understand division, especially those struggling with the concept of remainders. Ivan did not have a clear understanding of division, particularly division with remainders, prior to using the division template. When introduced to the tiles version of the division template, he began using that version almost exclusively. He said that he liked the tiles because "tiles help you because it shows you what's left over. … It helps you understand division." He was able to quickly create representations using a template and tiles. An example of his using the tiles is shown in Figure 9. In subsequent problems, he was able to use the division template alone to create correct representations and find solutions.



a Student creates a division template for 45 ÷ 9, then 9x2 array; 18 tiles for the array turn black

b Full division template; student solved the problem by snapping in 9x2 and 9x1 arrays

Fig. 9 A student uses tiles to help solve the division problem 45 ÷ 9

Using Structure and Organization

The division tool gave students a starting point when working problems and helped them better organize their work. Ariel's math work on pencil and paper tended to be disorganized, and her solutions had many mistakes that appeared to be careless. When using the division tool, however, she created very organized representations; both she and her teacher remarked on the difference the tool made. An example of her work is shown in Figure 10a. Replaying the *interaction history*, i.e., the sequence of actions the student performed in creating representations, shows that she worked quickly and easily with no false starts. In addition, her annotation of "tada, as easy as pie" shows a confidence that her teacher says was lacking from earlier lessons. For this student, as well as others, the teacher also noted that the tool slowed kids down and reduced careless errors.

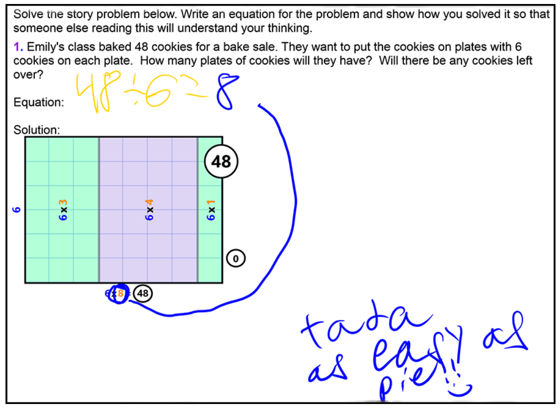


Fig. 10 Student example showing organized answer and confidence

Working With Independence and Confidence

The division tool also helped some students work more independently and confidently than when they solved math problems using pencil and paper. Kaitlin was below grade level in her math skills, and the teacher typically gave her problems with smaller numbers than the rest of the class got. Kaitlin also was anxious about math and had difficulty working independently. When using the tablets, and the division tool in particular, the teachers working with her noted that her anxiety was greatly reduced, she could work independently, and she no longer needed to be given smaller numbers. Prior to working with the division tool, Kaitlin had limited knowledge of division. While using the tool, she was able to start and complete division problems without assistance. She also contributed to class discussion, which the teacher reported was a rare occurrence. An example of her work with the division tool is shown in Figure 11. Replay of the interaction history for this example shows that she completed the work quickly.

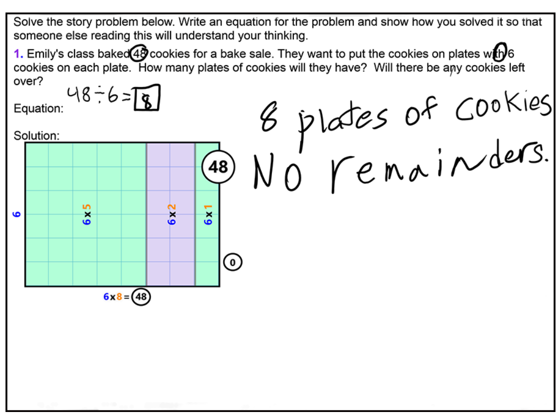


Fig. 11 Student example worked independently and with confidence

Creating a Variety of Representations

The examples shown in Figures 10 and 11 illustrate the tool's support for a variety of problem-solving strategies. The student work in Figure 10 shows solving 48 ÷ 6 with arrays of 6x3, 6x4, and 6x1; the work in Figure 11 shows 6x5, 6x2, and 6x1. Other students chose 6x4 and 6x4, 6x2, 6x3, and 6x3, or 6x7 and 6x1. These variations in student work provide useful fodder for class discussions of different approaches to division, with an emphasis on efficiency ("How could you do this problem using fewer arrays?") and familiar number relationships ("What multiplication fact could you use to get started on this problem?").

5 Current and Future Work

We are continuing our data analysis, focusing on evidence of the affordances and effects of the division tool on student's understanding of division. In addition, we are continuing our work on developing and testing automatic analysis routines that will enable the software to interpret student's representations [5]. Such interpretation enables students' thinking to be made visible to the teacher, who can more easily identify students who need help or choose student work for public display as a basis for class discussion. The analysis routines operate on both a representation and a student's interaction history, which captures the process of creating the representation. By analyzing the sequence of actions in creating each of the correct division templates shown in Figure 12, for example, the analysis routines report that the student whose work is shown in Figure 12a had no extraneous steps, which would indicate that the student understood how to use the tool to solve the problem. For the example in Figure 12b, the analysis routines report that the student had a false start (creating a template then deleting it, then creating another one), created arrays that were too large to snap in, and tried many times to add an array when the remainder was smaller than the divisor—an action that, through observation and interview, can be identified as indicating a misunderstanding of remainders. We are studying our corpus of student work in order to identify additional sequences of actions that will provide insight into what students do and do not understand about division.

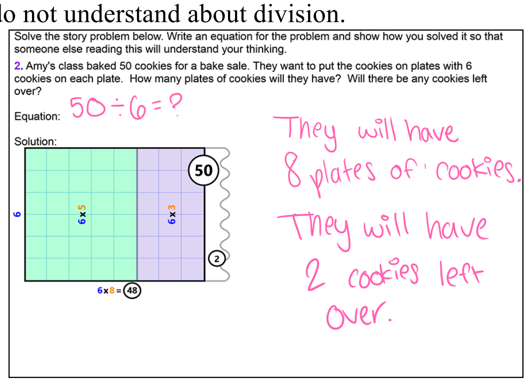


Fig. 12a Automatic analysis indicates that the student had no trouble

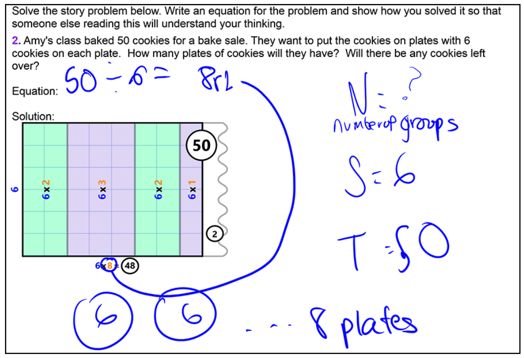


Fig. 12b Automatic analysis indicates that the student had trouble with the concept of remainder

We have prototyped a new division tool to be used with younger students who have not yet been introduced to arrays. The new tool is based on CLP's stamp representation [5, 7] and enables students to create a pictorial representation of partitive and quotative division problems and use either sharing or grouping strategies, respectively. It also keeps track of relevant numbers in the problem. A student starts by drawing an image to represent an object and another image to represent a group. (See the images in the boxes at the top of Figures 13a and 13b.) She copies the object image, using a "+" operator, to make a "pile" containing the number of objects specified in the problem. If the number of groups is given, and the size of group is unknown, she uses a sharing strategy: She creates the given number of groups, using the "+" operator on the group image and drags all the objects onto the groups. (See Figure 13a.) She is finished when all objects are in groups and each group contains the same number of objects. CLP keeps track of and reports the number of objects in each group below each group image. If the size of a group is given instead, she uses a grouping strategy: She creates one group object, drags the number of specified objects into the group object, then duplicates the group until all objects have been moved from the pile into groups. (See Figure 13b.)

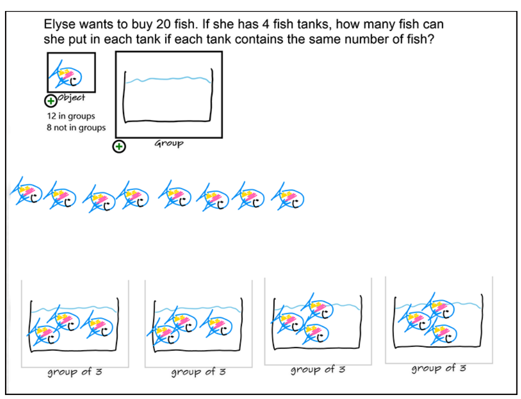


Fig. 13a Dealing out: 20 objects and 4 groups created; 12 objects dealt into groups, 8 remain

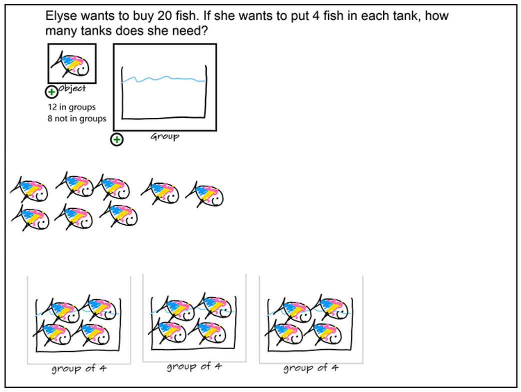


Fig. 13b Grouping: 20 objects and 1 group created; 4 objects moved into the group; duplicating the group moves 8 objects into new groups

We plan to study how 2nd graders use this new stamp-based division tool and to continue studying how 3rd and 4th graders use the array-based tool. Such research will contribute significantly to our continued investigation of the role tablet-based technology can play in deepening students' conceptual understanding of division.

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