

## ANIMATED MATHEMATICAL PROOFS IN ELEMENTARY EDUCATION Emerging Technology Research Strand

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### 1. ABSTRACT

The ability to prepare and present a mathematical argument, or proof, is a key component of the mathematical competence students need to achieve in elementary school. A proof for elementary students is not the highly structured deductive mathematical argument seen in high school algebra classes. Elementary students can, however, create mathematical arguments about equivalence using vocabulary appropriate for their level of understanding. The goal of the *Technology to Support Mathematical Argumentation* project is to develop computational tools with which elementary students can construct and share mathematical arguments. This paper reports on the development of array manipulation and animation creation tools that are extensions to our tablet-based Classroom Learning Partner (CLP) software. It also describes our experience in a Boston third grade classroom in which students were able to successfully create animations to demonstrate mathematical arguments about equivalence.

### 2. PROBLEM STATEMENT AND CONTEXT

The ability to prepare and present a mathematical argument, or proof, is a key component of the mathematical competence students need to achieve in elementary school. Developing this skill serves three distinct purposes for elementary math students: It supports and scaffolds their learning of computation, it prepares them for upcoming courses in algebra by introducing them to algebraic reasoning, and it begins the process of teaching them how to formulate and justify claims. Learning how to construct persuasive mathematical arguments can be challenging, however, and many students struggle with the subject matter. We believe that appropriate computational tools can help students develop their ability to fashion convincing proofs. In the NSF-funded *Technology to Support Mathematical Argumentation* project we are collaborating with leading mathematics educators who study early algebraic reasoning in order to design and implement computational tools with which students can construct and share mathematical proofs. These tools are being developed and tested within an existing tablet-based software system called Classroom Learning Partner (CLP), which supports the creation and sharing of student work as a basis for class discussion [1, 2, 3]. With CLP, students use the tablet computer's pen to interact with computational tools and to create inscriptions—handwritten sketches, notes, etc.—in an electronic notebook. Students share their work by means of the tablet computers' wireless networking. The teacher views and projects student work, using it to guide classroom discussion about alternate representations and problem-solving strategies.

Key to our development and research is the work of Russell, Schifter, and Bastable [4] and Schifter [5], which explored the idea of representation-based proofs, i.e., proofs about the behavior of arithmetic operations that employ representations of quantities rather than algebraic symbols. Many of the representation-based proofs described by these authors involve an initial representation of a quantity or quantities, a manipulation of that representation, and a final state. The “proof” is contained in the observation that, even though the structure of the representation has changed, the quantity represented in the initial and final state has remained the same. That is, there is a story to the proof. As one of the students in Schifter's work says, “It's like you

replay it in your mind, and now it makes sense.” [5, p. 79] One of the affordances of the pen-based tablet technology we are exploiting is the ability to record, save, and replay a series of actions on a representation, thus creating a *dynamic* representation.

We are designing the new computational tools by engaging mathematics educators who have studied how students develop mathematical argumentation skills, but who have little experience using technology to support mathematical thinking. To date, we have designed and implemented tools that allow students to create and communicate mathematical arguments about numbers and operations. We have identified the following design goals for these tools: (1) offer enough flexibility for students to create *novel representations*, (2) support creation of not only *static* representations but also *dynamic representations* that illustrate process as part of the argument, and (3) facilitate recording *explanations*, either oral or written, to accompany the representations. Below we describe the tools we have developed, our experience with them in a third grade classroom, reflections on our design criteria, and plans for next steps.

### 3. METHOD EMPLOYED

#### 3.1 Classroom Setup

In a CLP classroom, students and teacher each have a tablet computer, and a tablet is connected to a projector. At the start of a lesson, students log in and are automatically connected to a wireless peer-to-peer network. They load the lesson’s electronic notebook, which the teacher has created using an authoring system. Students work through the lesson—individually, in groups, or as a class—and wirelessly submit their responses, which appear on the teacher’s machine. The teacher views student work while circulating through the classroom, identifying students who need help. She also conducts class discussion focused on student work examples that she has selected and sent wirelessly to the projector to be displayed anonymously. The teacher’s UI is shown in Figure 1. The student’s UI is shown in Figure 2. In both versions of the UI, users select commands by tapping on icons on the top command bar with pen or finger. Thumbnails of notebook pages are shown in the panel on the left, which the user can hide from view. In addition, the teacher can view student work for a particular page by tapping on the person tab associated with a page. By tapping on icons below each page of student work, the teacher can tag work so that it can be sorted by those tags. In addition, the teacher can create and project displays showing multiple examples of student work at the same time. The student version has a *Send to Teacher* button for submitting notebook pages to the teacher.

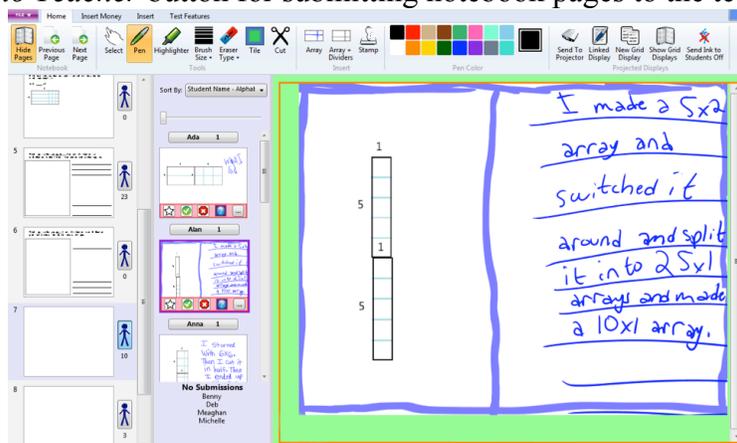


Figure 1. Teacher UI: thumbnails of pages are on the left, student submissions for a page are to the right and accessed via the person tab for the page, student work is displayed in the main window

### 3.2 Software Development

To support teaching and learning of mathematical proofs, we added two new features to CLP: a new math tool and an animation tool.

New Math Tool. The new *Cut* tool enables students to split CLP's mathematical representations, e.g., an array, into two objects. Such a cutting tool is necessary for the kinds of mathematical proofs that are prevalent at the elementary grade level, e.g., showing mathematical equivalencies such as  $4 \times 6 = 8 \times 3$ . For such a proof, a student could create a  $4 \times 6$  array, cut it into two  $4 \times 3$  arrays, then move one of the  $4 \times 3$  arrays below the other one to show an  $8 \times 3$  array. Shown in Figures 2a and 2b are the beginning steps in such a proof. To use the *Cut* tool, a student taps on the ✂ icon in the top command bar and uses the tablet pen to draw a continuous stroke across the object where he or she wanted to split it, as shown below. While the line is being drawn, the cursor is a scissors icon.

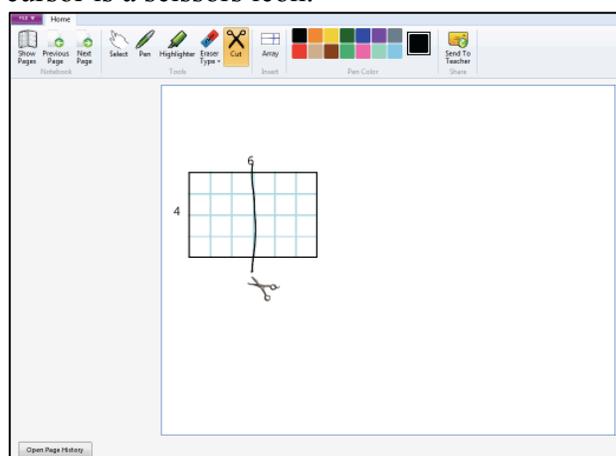
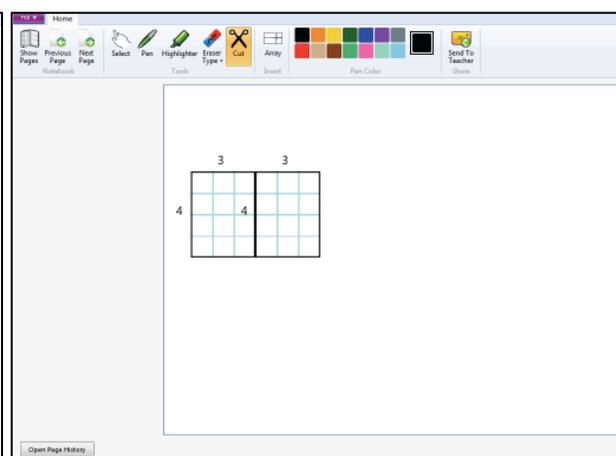


Figure 2a. To cut an array, student taps on the *Cut* tool then draws a line at a cut location



2b. Result of cutting an array is two new arrays, each of which can be moved independently

Animation Tool. To enable elementary students to create a dynamic representation that illustrates a mathematical proof, we designed and implemented a simple animation tool. With such a tool, students have an artifact of their work that can be easily replayed and shared with other students and the teacher. Having an animation also avoids a problem with paper-based animation proofs: Having separate representations of the initial and final state of a proof represents the quantity in question twice, rather than having a single representation that is modified.

In order to avoid having animation controls on the command bar, where they would be available on every page and might prove distracting to students, we created a new kind of CLP notebook page with animation controls at the bottom of that page, out of students' way. The new Animation Page with controls is shown in Figure 3.

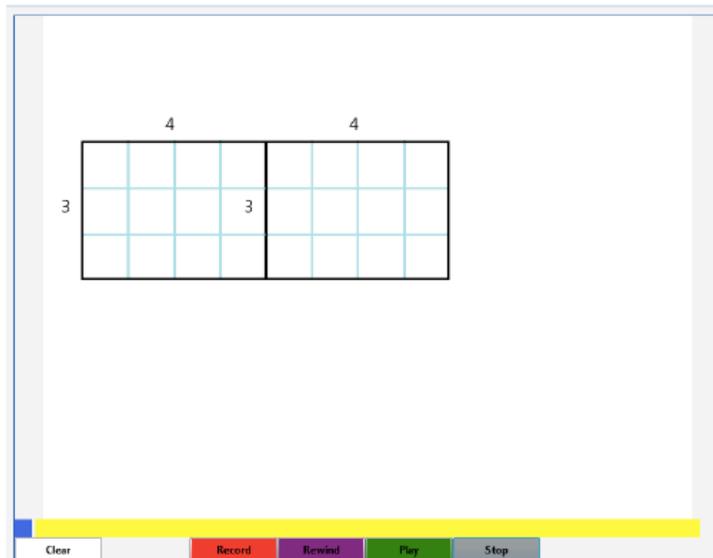


Figure 3. CLP page for creating and playing animations using controls at the bottom of the page

In designing the animation controls, we wanted to have a UI that was intuitive and easy for students to use, so we patterned our UI after that of a general media player, which students are likely to have seen before. As shown in Figure 3, it has a *Record* button that is used to indicate when actions would begin to be recorded to create an animation. It also has *Rewind*, *Play*, and *Stop* buttons for the animation. To create an animation, a user taps the *Record* button, manipulates objects on the notebook page, then taps *Stop*. Tapping *Rewind* then *Play* replays the animation. Stopping an animation at any point during playback then tapping *Record* records over any animation that follows the stopping point.

We included indicators to give the user feedback about the presence and state of an animation: An *Animation Present* indicator—a small box on the bottom left of the animation page—is white when no animation is present on the page and blue when an animation is present. An *Animation Progress* bar is a yellow bar that spans the bottom of the page, indicating the user’s location in the animation, i.e., how much of an animation has been played back. An initial design proved to have buttons that were too small for students to easily tap, so the buttons were enlarged to the size shown in Figure 3. The buttons also are brightly colored in order to be easily distinguished from one another and to help make the UI appealing to elementary students. We included a *Clear* button after observing in an initial design that students often wanted to try out several versions of an animation before finalizing one. The *Clear* button is spaced away from the other control buttons in order to decrease the chance of accidental tapping. Finally, after observing that students often wrote their explanations while recording, we made contiguous text appear on the screen at once during playback. Without this feature, when a teacher replays a student’s animation for the class to see, significant time may be spent watching a student’s ink appear and disappear if the student erased it.

### 3.3 Classroom Trials

Our classroom observation of students using the new CLP features took place in two Boston-area classrooms—four days in a classroom with 25 third grade students and two days in a classroom with five fourth and fifth grade special needs students. Described here is our experience in the classroom of 25 students.

The classroom was at the Mattahunt School in Boston, MA, an inner-city school with students of predominantly African-American, Latino, and Caribbean backgrounds. We spent four days in this classroom with class durations between 60 and 90 minutes. The class had participated in the NSF-funded project *Using Routines as an Instructional Tool for Developing Students' Conceptions of Proof*, led by Russell, Schifter and Bastable, so the students were familiar with the process of using representations to prove generalizations about arithmetic operations. On the first day, students spent 15 minutes getting used to navigating through pages in a CLP notebook, inking, submitting work to the teacher, and creating arrays. They spent the remainder of the class using CLP to create and discuss representations that employed arrays to visualize multiplicative relationships, e.g., showing all the ways to use arrays to represent the number 32. On the second day, students used array creation, cutting, and animation tools to divide large arrays into smaller ones and to discuss the mathematical relationships between the large and small arrays. On the third and fourth days, the students worked on proofs that required them to coordinate use of all the tools they had employed on the previous days. The student work discussed below is from the fourth day.

Problem 1: How can you prove that  $3 \times 4 = 6 \times 2$ ?

In order to solve this problem, students had to create either a  $3 \times 4$  array or a  $6 \times 2$  array, cut it in half and re-arrange the halves to create the other array, recording the transition from one shape to the other in an animation. The proof of the equality of the two expressions is inherent in the fact that, while the shape of the array has changed, the representation still contains the same number of units. Several student responses are shown below. AL's response, shown in Figures 4a and 4b, is typical. She began with a  $3 \times 4$  array, cut it into two  $3 \times 2$  arrays and moved one of the smaller arrays under the other small array to create a  $6 \times 2$ . Her explanation reads, "I started with a  $3 \times 4$  array, then I made it into a  $6 \times 2$  array. I cut the  $3 \times 4$  array into a  $3 \times 2 +$  a  $3 \times 2$  array, then move one  $3 \times 2$  array under the other one."

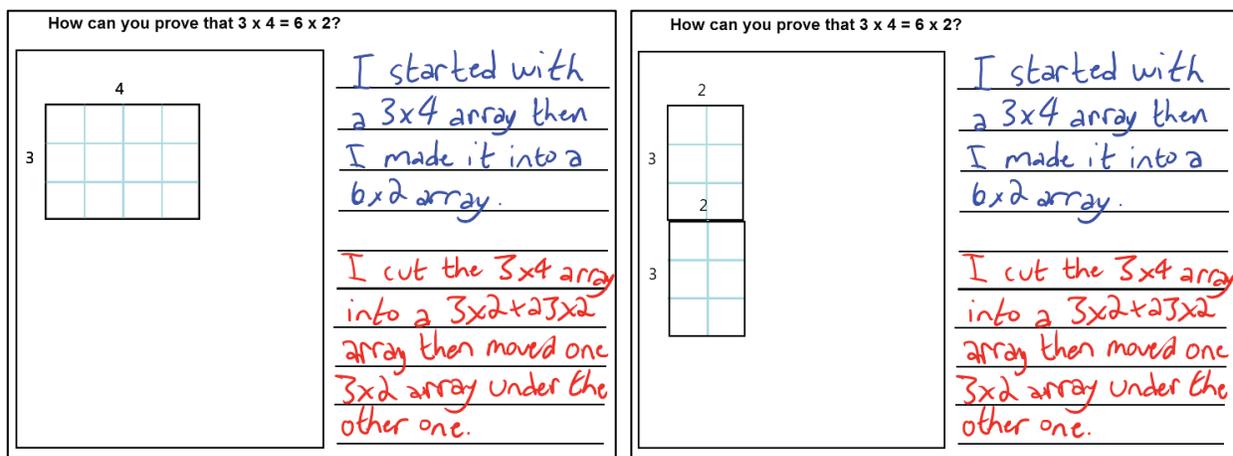


Figure 4a. AL's proof for Problem 1: Beginning 4b. Ending

Some students wanted to show the array transformation in both directions, i.e., from  $3 \times 4$  to  $6 \times 2$  and from  $6 \times 2$  to  $3 \times 4$ . One example is DO's work shown in Figures 5a and 5b. Before starting her recording, she created a  $3 \times 4$  array, a  $2 \times 6$  array and two  $3 \times 2$  arrays. Her final screen shows just two arrangements: a  $6 \times 2$  made up of two  $3 \times 2$  arrays and a  $3 \times 4$  made up of two  $3 \times 2$  arrays. Her written explanation—"I did a  $3 \times 4$  then I move the cube under the other cube now

it's a  $3 \times 4$ . Now I made a  $6 \times 2$ . I cut it in the middle and put to a long stick.”—is a little hard to follow, partly because of her use of some vocabulary that had been developed in her classroom, e.g. “stick” to mean arrays that are much longer than wide. Her oral explanation in class, however, was quite clear.

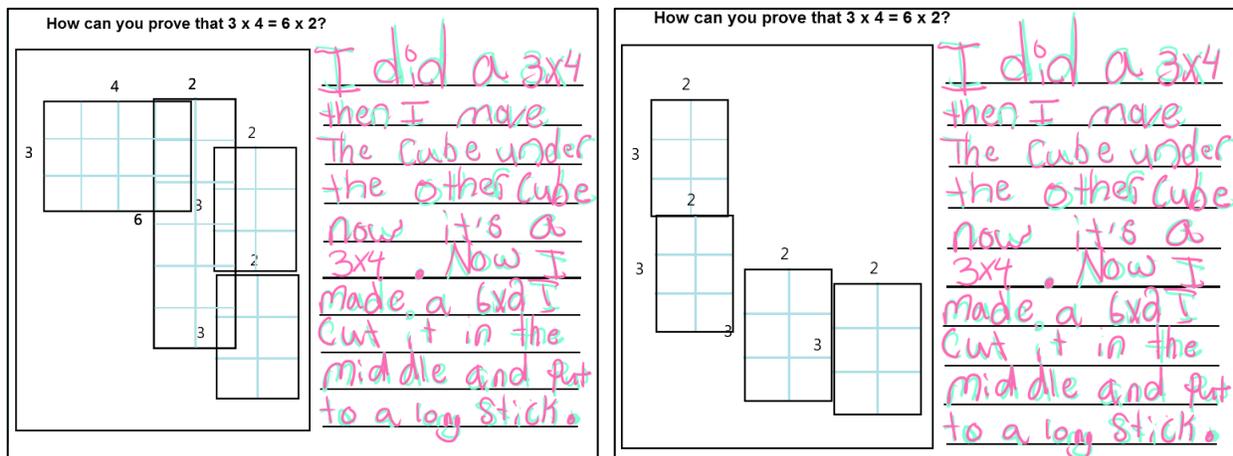


Figure 5a. DO's proof for Problem 1: Beginning 5b. Ending

While few students actually wrote down a statement that “the quantity had not changed, just the arrangement of the arrays,” the teacher led the class discussion in that direction and at least one student included the following in his work, “The arrays are different sizes but have equal numbers.” (By “size,” he means “shape.”)

All but three students were successful in creating animations to prove this equivalence. (One student with behavioral problems did not want to participate; a second student could create animations but did not understand the math; a third student showed that  $3 \times 4 = 12 \times 1$ , rather than  $6 \times 2$ .) Several students went on to make up their own similar problems, creating animations showing, for example, that  $5 \times 2 = 10 \times 1$ , or  $6 \times 6 = 12 \times 3$ .

**Problem 2:** What happens when you subtract 1 from one of the factors in a multiplication problem? Create an animation to show what happens.

While this problem was stated as a general question, we decided in consultation with the teacher to pose it about a particular multiplication problem, namely  $6 \times 4$ . We planned to move to the more general problem statement if we had time, but we did not.

The key insight we hoped students would have in working on this problem is that subtracting 1 from one of the factors can be modeled by subtracting a row or a column from the array representing the original multiplication problem. Some students initially thought that they could model “subtracting 1 from a factor” by subtracting 1 from the product. In fact, for better or for worse, our software made this impossible, since using the *Cut* tool on an array automatically cut off some number of rows or columns. However, we observed at least one student (JA) trying—unsuccessfully—to cut off a single unit from an array. Her eventual solution to the problem is presented in Figures 6a and 6b, which show the beginning and end of the animation she created. She cut off one column from the  $6 \times 4$  array and moved it to the side. Her written explanation offers a window into her struggles: “I notice that if you take away 1 your answer would be 1 more less and if you take away 4 your answer would be 4 more less.”

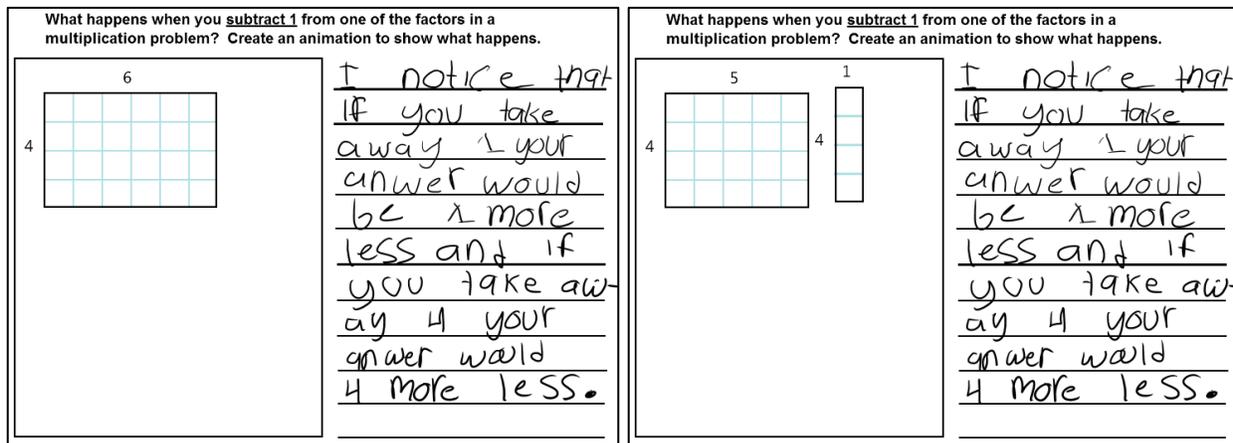


Figure 6a. JA's proof for Problem 2: Beginning

6b. Ending

While many of the students created animations similar to JA's, cutting off and moving either a row or a column, several students saw an additional level of complexity in the problem and realized that they could subtract one from *either* factor, ending up with two different array manipulations. ME, for example, created two 6x4 arrays, cut a row off one and a column off the other and wrote, "I started with 6x4. I cut one of sides of 6x4 and made it in to a 5x4. When I cut the other one I took the top one and it made 6x3." Starting and ending states of his animation are shown in Figures 7a and 7b.

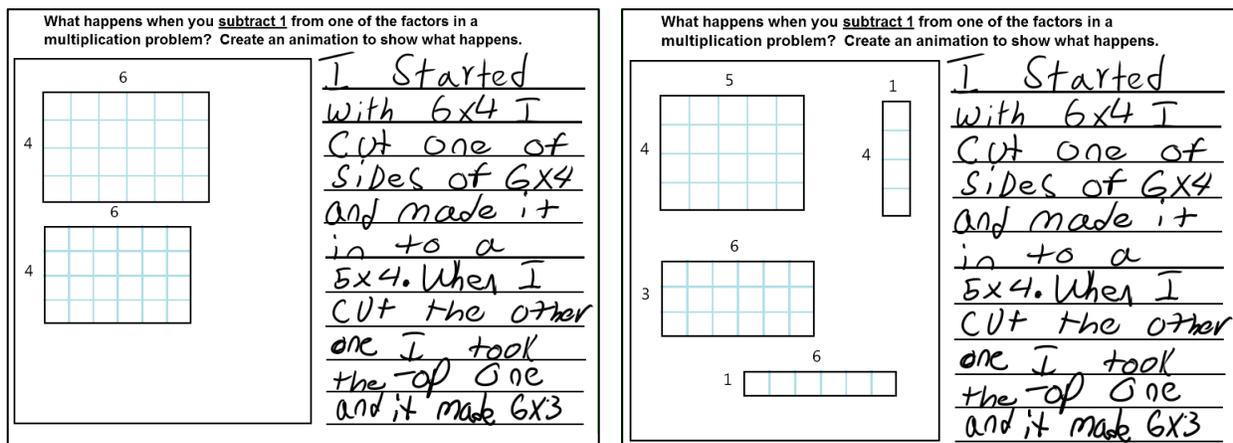


Figure 7a. ME's proof for Problem 2: Beginning

7b. Ending

Another student added a dramatic flair to his solution by cutting off a row, moving it away from the original array, then deleting it. During the class, he was adamant that we display his work on the projector, so in the last few minutes of class we did—and the disappearance of the cut-off row was greeted with appreciative laughter from the other students and the teacher.

All but four students were able to create animations for this proof. (One again did not want to participate; one forgot to tap *Record*, but was nonetheless successful in cutting and moving arrays to demonstrate the proof; two did not understand the math, though they could create animations.)

#### 4. RESULTS AND EVALUATION

Our reflections are organized around our three design goals: *novel representations*, *dynamic representations*, and *explanations*.

- Were students able to create *novel representations*?

Students were able to use the tools quite facily (and enthusiastically), but their solutions had only minimal variability, due to the narrow range of tools that we provided them. There was more variability in their solutions to the second problem—subtracting 1 from a factor—but even there, most solutions were similar. Our next steps in providing students with more expressive options will be to integrate more tools with the animation capability and to develop a fuller range of tools to construct and manipulate arrays.

Adding array construction tools to the set of tools students have available may make it possible for students to express more “wrong” ideas, e.g., cutting off a single unit from an array, rather than cutting off a row or column, as JA wanted to do. There is an interesting educational dilemma encapsulated in this design question: Some digital tools lead students to “correct” ideas but may short circuit the re-conception process that may ultimately lead to deeper understanding. In this case, if JA had been able to just take off one unit from her array to show her understanding of “subtracting 1 from a factor”, which she would have been able to do with physical cubes, the classroom conversation would have been different. Would this have been “better” for her learning? Or for the class as a whole? Having more flexible tools will allow us to examine this question in more depth.

- Were students able to create *dynamic representations*?

The animation tools worked well, and students learned to use them quickly. While we do not yet have direct comparisons, our impression is that these tools enabled students to express mathematical equivalences more easily than they could with pencil and paper, physical manipulatives, or static drawing tools. As the teacher said, “On paper they draw the beginning and the end. It’s so much easier to see the proof when the kids can create the animation.”

Our classroom observations led to several improvements in our UI design, which are described in the next section. In particular, we noticed that the teacher and students would sometimes want to forward/rewind an animation to a specific spot, and sometimes the students lost track of what they were doing with an animation (rewinding, editing, replaying, etc.). In addition, the teacher sometimes wanted to vary the speed at which an animation played back. For more complex animations, the teacher would want to play them back slowly so that students could see exactly what was taking place, but for simpler animations, she wanted to play them back more quickly.

- Did the tools facilitate both written and oral *explanations*?

We had originally hypothesized that adding the ability for students to record an audio explanation to accompany an animation would enhance their ability to explain their thinking, but we did not implement such a feature for two reasons. First, we realized in the design phase that it would be difficult for students to synchronize their verbal explanations with the animation. In general, the animations for the number proofs in our curriculum are brief and last a far shorter time than it takes for a student to describe what is happening in the animation. As described above, the issue of playback speed was a complicating factor, and we decided that integrating audio at this point would be more of a burden than a help to the users. Second, we had discovered in previous experiments with audio explanations that it was quite difficult for an entire class to hear an explanation played on a student’s or teacher’s machine. In our trials with

the animation tool, the students' written explanations were sufficient to support oral class-wide discussions about their proofs.

There are students, however, who find writing daunting, and they may benefit from an audio recording capability. Such audio recordings may be especially helpful when a teacher will be reviewing work outside of class when students are not available to describe or answer questions about their animations.

## 5. CURRENT AND FUTURE WORK

The current version of the animation software has three new features, the first two of which are shown in Figure 8.

Location Slider. We added a location slider, overlaid on the animation progress bar, that allows a user to play an animation by moving the slider back and forth, until she gets to a specific location. The slider allows a user to forward an animation past uninteresting or repetitious sections.

Variable Playback Speed. Initially we attempted to decide on an optimal average speed for animation playback, but it is better to create a user control, implemented as a slider, that allows the teacher and students to change in real time the speed at which an animation is playing. The slider is located between the *Clear* button and the animation control buttons.

Recording Cue. We made the border of the animation area flash red when users are recording an animation, providing an extra cue to help users remember to press the *Record* button.

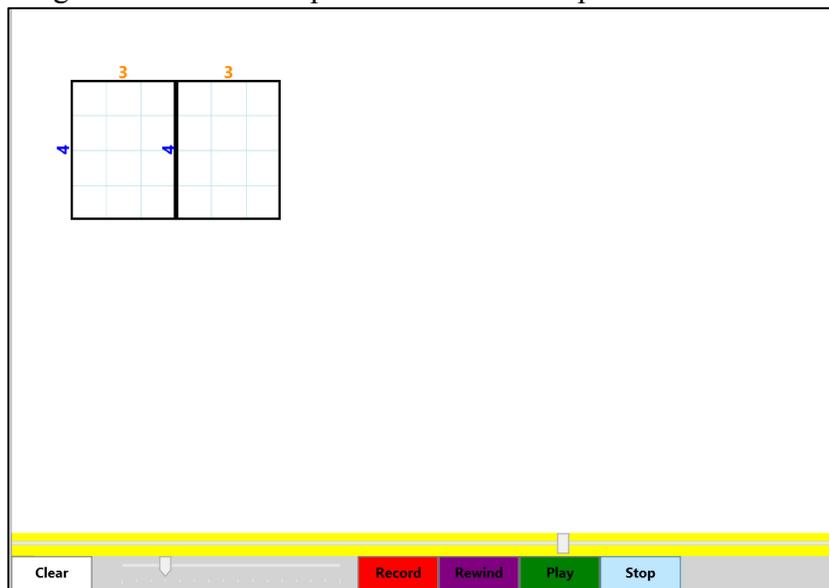


Figure 8. New features: Location slider (on yellow progress bar) and variable playback speed (below progress bar)

In addition to modifying the animation tool, our current and future work involves making our array tools more flexible, as discussed in the previous section, e.g., to enable students to express more “wrong” ideas such as removing a single unit from an array rather than a row or column. It also involves providing students with tools to prove generalizations about arithmetic operations rather than specific instantiations of generalizations. While Problem 2, for example, was phrased in full generality—“What happens when you subtract 1 from one factor in a multiplication problem?”—students actually worked on the problem in the context of  $6 \times 4$ .

While they were successful in their use of the technology and cogent in their arguments about the specific problem, there was no obvious way for them to use the technology to make the general argument because they could only create specific arrays using our tools to specify the numbers of rows and columns. In order to articulate general proofs, elementary students need a representation that achieves what “x” and “y” do in formal algebraic proofs—in this case, a representation that enables students to indicate that the actual numbers of rows and columns do not matter. Using such a representation, students then could state proofs about the structure of numbers and the effect of manipulations on that structure. One such representation can be modeled after what students do when using physical manipulatives such as Unifix cubes to state a proof about numbers: The Boston teacher reported that when her students talk about proofs, they sometimes use their hands to cover part of their representation in order to indicate that it doesn’t matter how many cubes are in the “covered” part of the number. We are designing for CLP a similar software “hand” that will allow students to cover parts of an array to distinguish between the critical parts of the structure of a number, i.e., those parts that are important to the proof, and those indeterminate parts that are “variable.” We plan to observe the use of this new feature in our next classroom trials.

## 6. ACKNOWLEDGMENTS

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