

6.UAP Final Report:

**Machine Interpretation of  
Elementary Math Students'  
Hand-drawn Bin Representations**

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# 1 Introduction

For the past eight years, Dr. Koile and her team have worked on the NSF-funded project *INK-12: Teaching and Learning Using Interactive Ink Inscriptions in K-12* (<http://ink-12.mit.edu>), which investigates improvements in K-12 math education through tablet technology [2-5]. To support their investigation, they developed and tested a pen-based classroom interaction system called Classroom Learning Partner (CLP). CLP allows teachers to display exercises in an electronic notebook on students' tablets, where students then can use a combination of freehand drawing and various digital tools to create and interact with a visual representation to solve math problems on "pages" in their notebooks. CLP attempts to understand the student's work and shares with the teacher a concise view of any items that might reveal the student's problem-solving process.

## 2 Background

A common strategy students use for division is the so-called "dealing out" method [1]: Given a dividend, which represents a total number of objects, and a divisor, representing a certain number of groups, students can use physical manipulatives to create piles of equal groups, dealing out objects one or several at a time until all objects are in groups. The number of objects in each group then represents the quotient.<sup>1</sup> When using a drawing tool, whether a pencil or a tablet pen, a similar activity involves a student drawing enclosed shapes we call bins and filling the bins with marks until a total number of marks has been added. In the example shown in Figure 1, a third grade student solves  $48 \div 8$  by drawing 8 circles representing bins and placing marks in each of the bins until she has placed all 48 marks in bins. At the end of the process, the answer is represented by the 6 marks in each bin.

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<sup>1</sup> A division problem in which the number of groups is given is known as a partitive problem. A division problem with the size of group given is a quotative problem.

10. Amelia makes 48 paper cranes for her teachers.  
 She gives each teacher a string of 8 paper cranes.  
 How many teachers does she give paper cranes to?

6 teachers

$$48 \div 8 = 6$$

Figure 1: Third grade student's bin representation for  $48 \div 8$

After observing such hand-drawn bin representations in a local third grade classroom, the INK-12 team designed and implemented a structured digital Bin tool that facilitated the dealing out process by helping students keep track of the number of marks in each bin. The goal was to free students from the task of keeping track of the total number of marks created, since students often lost track of their count and had to start over in their dealing out process. Shown in Figure 2 is an example of CLP's Bin tool being used by a student to solve  $64 \div 4$ .

Try the new Bin tool for solving this problem.

$64 \div 4 =$  16

**64 in Bins**  
**0 not in Bins**

Figure 2: Third grade student's representation using CLP's bin tool for  $64 \div 4$

MIT student Dai Yang worked on a previous UAP project [6] that allowed CLP to interpret a representation created using the CLP Bin tool and the associated dealing out method. As a result, CLP now can provide the following relevant information to the teacher: The student drew 4 bins, dealt out 8 dots at a time into each bin, and ended with 16 dots in each bin. The key insight is that the student dealt out 8 dots at a time, showing that he might have had an intuition for the final answer before creating the representation.

It was not difficult to distinguish between marks placed inside and outside of a predefined square bin, as in Figure 2. The challenge lay in identifying and generalizing the pattern of addition of marks to the bins.

Students nevertheless may still choose to draw bins, so in this UAP project, we've extended the infrastructure created for machine interpretation of CLP's Bins in order to handle the situations in which students create hand-drawn bins, as shown in Figure 1 and Figure 3.

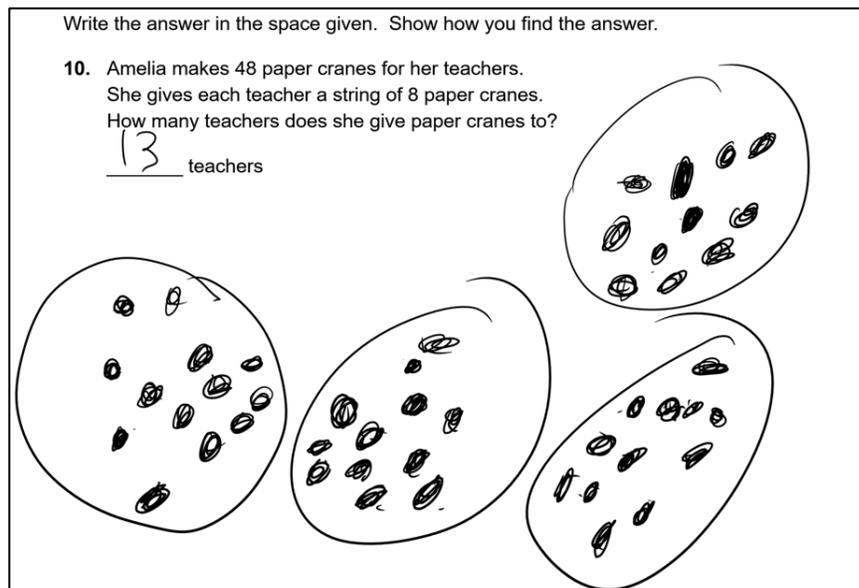


Figure 3: Third grade student's (incorrect) hand-drawn bin representation

### 3 Software Implementation

The crux of the work in this UAP project was coding CLP to identify hand-drawn bins and to determine the bin boundaries. The rest of the machine interpretation problem resolves to identifying the dealing out strategy as with CLP's structured Bins in Dai Yang's UAP project. The simplest case to start with is identifying a large enclosed shape made by a single ink stroke, such as those shown in Figures 1 and 3 and below in Figure 4a. Figure 4b shows CLP's identification of the enclosed shapes in Figure 4a; the creation of CLP's black boundaries for the shapes is discussed below.

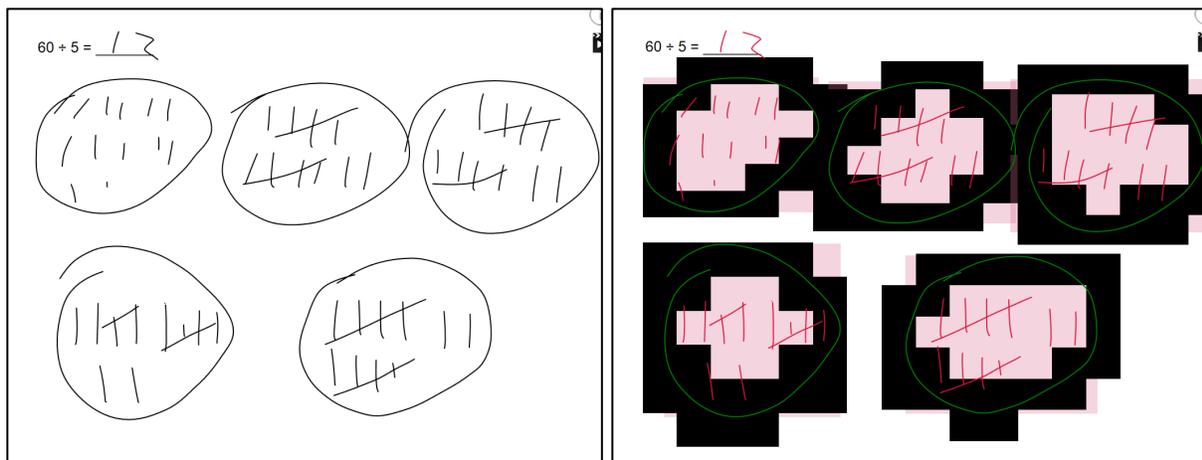


Figure 4 a. Example of hand-drawn bins

b. CLP's identification of bin boundaries

The strategy for identifying enclosed shapes is to break up the region surrounding a stroke into fixed-size, square “cells”. Then the code uses the points along the stroke to identify occupied cells, which are denoted by the black regions in Figure 4b. Finally, the code runs a modified cycle-detection algorithm, described below, to determine whether these occupied cells formed an enclosed shape. As shown in Figure 4b, all the strokes that are correctly identified as forming enclosed shapes turn green.

There were three reasons for this approach. First, it is relatively simple. There are machine vision and machine learning techniques that could solve the enclosed shape problem, but

that would be like using a sledgehammer to crack a nut. A major design guideline for CLP is that the simpler solution is preferable.

Second, the approach accurately captures the fuzziness of hand-drawn enclosures. One foreseen challenge was identifying bins such as shown in the bottom left of Figure 4a. This enclosure does not completely connect, but the ends are close enough for the intention to be clear. As long as the gap isn't too large, the occupied cells will be neighboring, and the enclosure will be correctly identified.

Third, the approach easily extends to identifying enclosures made by multiple strokes, such as the upper right bin in Figure 3: Careful inspection reveals that the student used two strokes to form this bin. Once the code can correctly group these two strokes, it can simply group the points along each stroke, and the problem resolves easily to the single stroke case.

The cycle-detection algorithm at its core is a depth-first search (DFS) that checks whether an enclosure can be made from any connected cycle of occupied cells. There were two major modifications made to reduce the number of false positives. First, whenever a cycle was detected via DFS, the code checks that the region enclosed by the cycle is large enough according to the cell bounds. If the region is too small, the DFS discounts this cycle and continues. The minimum size for an acceptable "enclosed region" scales with the size of the stroke itself. Second, the code uses a heuristic to prevent any search path from looping back towards itself. If the next cell along a path already has three neighbors on the path, it is discounted. This accounts for weird cases where the occupied cells form blocks that can be snaked through on the diagonals to form crisscross shaped paths. Both of these modifications are guided by the intuition that we would like the path along the detected cycle to roughly mimic the path of the stroke. If a cycle is detected that does not follow the path of the stroke, this could easily be a false positive.

In the first version of the implementation, we noticed that small strokes were being incorrectly identified as bins all the time because the strokes occupied a tight knit square of cells that would always count as a cycle. The fix was to make the cell bounds and threshold size for cycle region scale according to the size of the stroke region. We also noticed in the examples that we were using to guide the implementation that students didn't draw very oblong enclosures (width much greater than height or vice versa), so that information could be used to immediately rule out certain strokes before starting the bin identification algorithm.

## 4.1 Third Grade Student Results

Throughout this project, a set of 264 pages of student work from a local third grade class was used for testing the algorithms and producing interpretation results. The work was the final assessment for a five-week classroom trial in which CLP was used for instruction of a multiplication and division unit in a class of 22 third graders (22 students x 12 problems, 1 problem per page). For the unit, students generally used CLP's digital tools—array, number line, stamp—to create representations, annotating the resulting representations using digital ink. On 11 pages, however, students used hand-drawn bins in their representations. Ten of the pages, examples of which are shown in Figures 1, 3, 5, and 6 contained a total of 49 hand-drawn bins that were discrete shapes; one page contained seven bins drawn as a grid, machine interpretation of which was outside the scope of this project. All 11 pages of work with hand-drawn bins are shown in Appendix A.

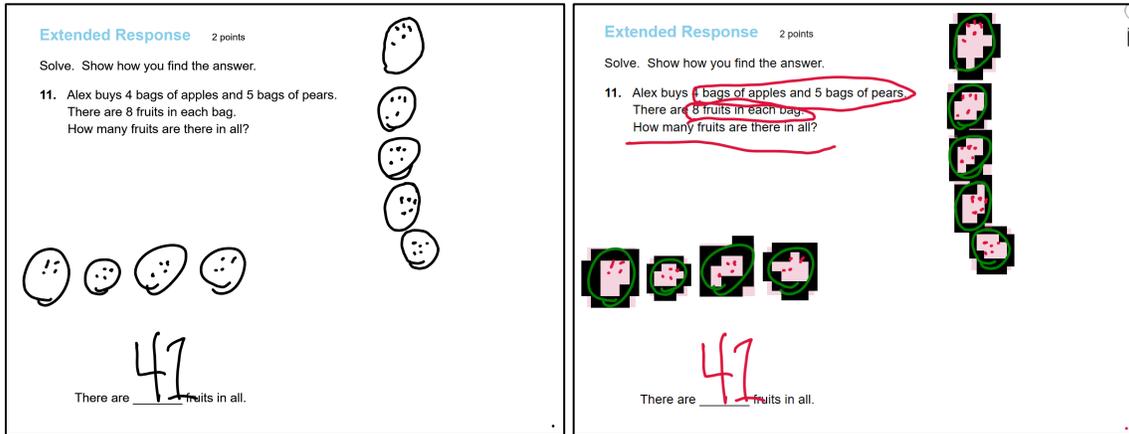


Figure 5. Additional example of third grade students' bin representations with black boundaries to show CLP's interpretation

The code was designed to work well with the student work, and it does. All 49 student hand-drawn bins were correctly identified. (See Appendix B for CLP's interpretation of the bins for the student work.) In one case a student used two strokes to form a bin, and the code correctly identifies the bin. The only errors occur when the loops in the numbers "8" and "6" are incorrectly identified as bins, as shown in Figure 6. Ruling out enclosing strokes such as these, which are not meant to be bins, is a difficult problem, but one that may be solved by using context (see discussion in the Future Work section).

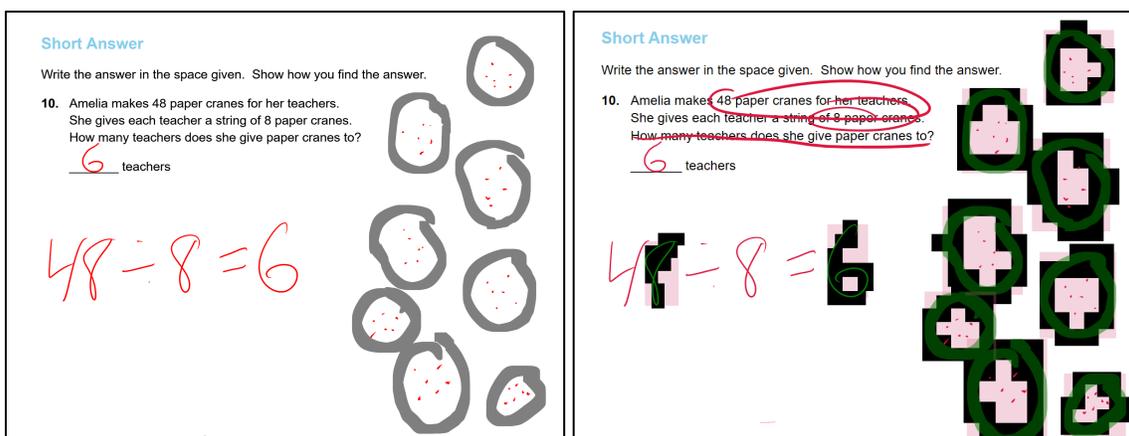


Figure 6. The numbers "6" and "8" are incorrectly identified as bins.

## 4.2 Adult Tester Results

In order to test the bin interpretation code on more than 8 examples, 10 adult testers were recruited to work division problems using hand-drawn bins and a dealing out method. The 10 volunteers solved five problems each, for a total of 50 more worked examples. The volunteers drew a total of 230 bins. The five problems and other examples are presented in Appendix C, and some examples are included below.

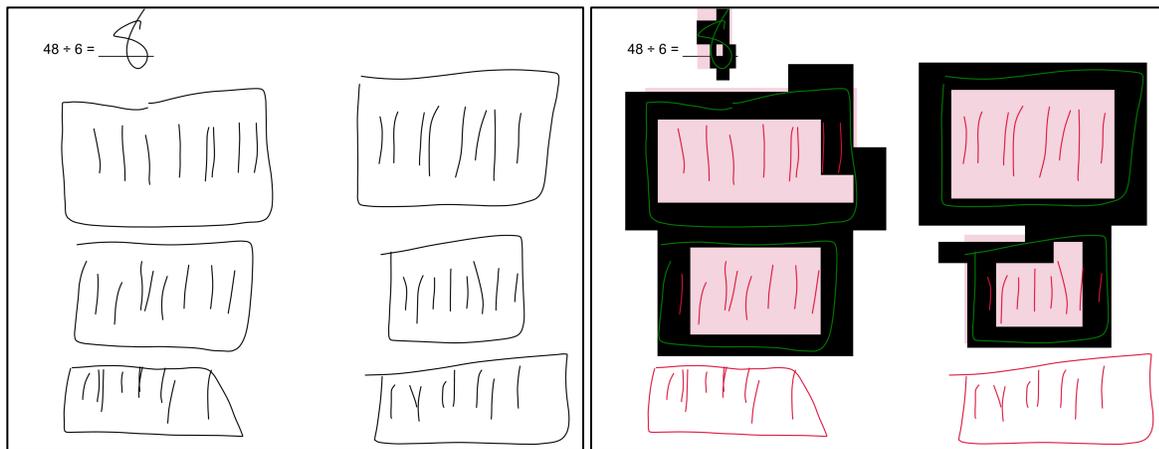


Figure 7. Two bins are missed because they are slightly too oblong, and the number “8” is incorrectly identified as a bin.

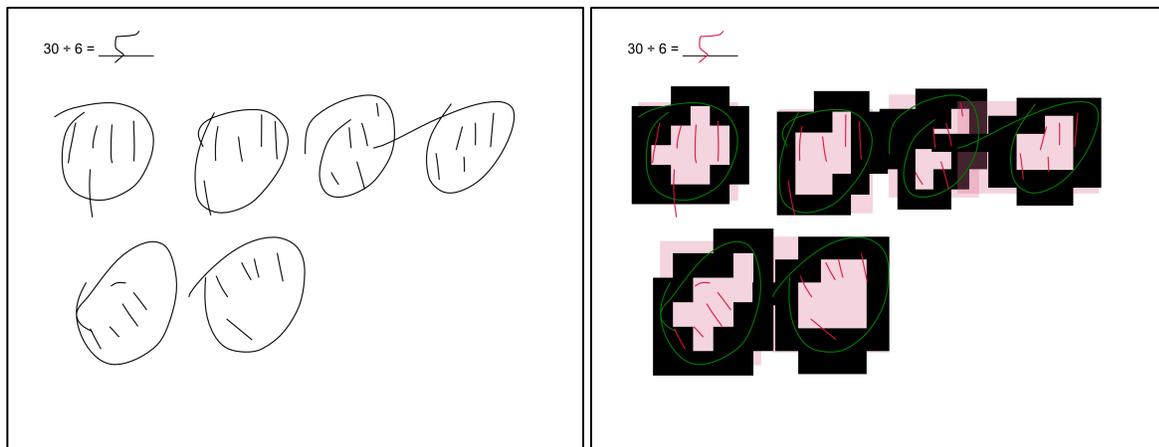


Figure 8. All six single stroke bins are correctly identified.

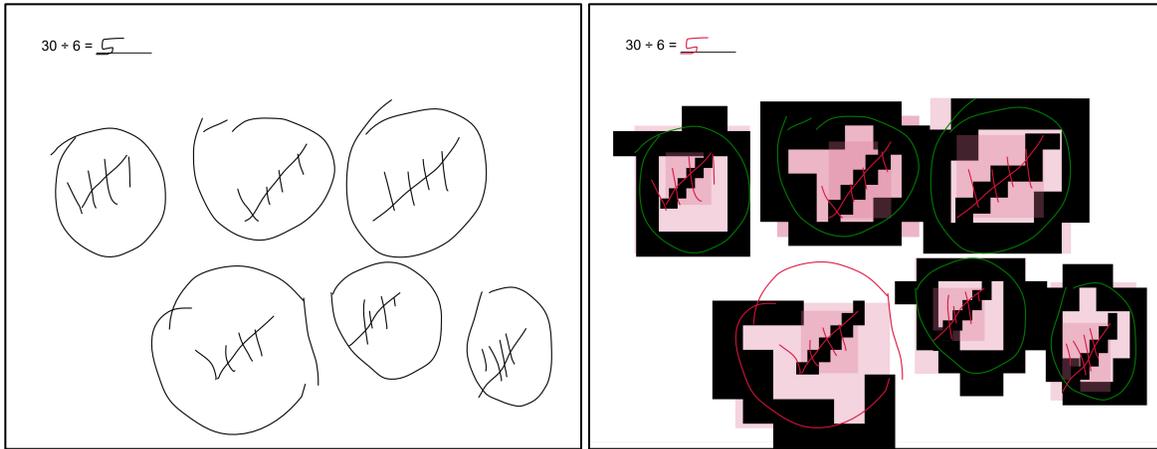


Figure 9. All four single stroke bins are correctly identified, one two stroke bin is correctly identified, and one two stroke bin is missed.

I often encouraged testers to draw bins poorly. Since all my testers were adults, my goals were to model kids’ interaction more accurately and to stress test the bin identification algorithm. These tests were useful for updating the constants used as acceptable thresholds, for instance smallest allowable bin size or the most oblong allowable shape. The results of testing are shown in Table 1.

Table 1. Test results for adult volunteers’ hand-drawn bins

	Successful	False Positive	False Negative
1 stroke	204	5	26
2 strokes	2	0	6
3+ strokes	0	0	10

The bin identification algorithm performed well for single stroke bin identification, even when the adult testers were encouraged to be “sloppy”: In the 50 problems, the algorithm correctly identified single stroke bins 204 times out of 230 (89%). It failed for 26 bins, 24 of which were very oblong bins. Those failures are easily correctable by either softening the

aspect ratio threshold or doing away with it altogether. Only twice were single stroke bins so ill-formed that the algorithm did not detect them.

There were 5 false positives where numbers were identified as bins, and those numbers were always “6” or “8”. This mimics the only error found in the third grade student results and is discussed in Future Work.

Two stroke bin identification was less successful but shows promise. When multiple strokes are correctly grouped, the identification is just as accurate as in the single stroke case. The problem lies in correctly grouping strokes meant to form an enclosure. The simple approach currently used is to only check for enclosure creation when the endpoints of two strokes meet, where the threshold for how close the endpoints need to be to ‘meet’ scaled with the size of the strokes. When the tester draws strokes past one another, strokes will not be grouped as they should. This approach succeeded two out of eight times, but the six failures all occurred on the same page and the tester was encouraged to draw these types of non-meeting strokes. Similarly, the 10 cases in which bins were drawn with three or more strokes were not identified. There is already existing infrastructure in CLP for k-means clustering of strokes that could be used to address this problem. This clustering approach would be applicable to grouping any number of multiple strokes, although it only occurred once in the third grade students’ work, so multiple strokes may prove to be a rare occurrence.

## 5 Future Work

As mentioned previously, one important improvement for identifying bins is to use context clues in order to reduce false positives, such as identifying a “6” as a bin.

One source of context clues would be the math in the problem: For example, if the problem is  $30 \div 6$ , and the algorithm “knows” to expect 6 bins, but identifies 7 bins, one of which

contains no marks, then that bin is likely a false positive. Another context clue might be provided by the set of bins drawn on the page: Perhaps the algorithm could reason that if a bin doesn't have roughly the same bounds as the other bins, then it may be a false positive. In addition, it seems likely that a student wouldn't draw an extra bin and put zero marks in it, so such a bin might be a false positive.

Context clues also may help with the problem of false negatives. If the problem is  $30 \div 6$ , for example, and the algorithm identifies only 5 bins, the code could re-run the identification algorithm with less strict bounds and also searching for interior marks.

In general, using context is an important extra input that will improve hand-drawn bin identification. Accounting for both the size of other bins and the math problem given on the page are the most important next steps in making this bin-identification algorithm more robust.

## 6 Conclusion

The goal of CLP is to explore possible improvements in math education by giving students better tools and presenting actionable information to teachers. Enabling CLP to interpret hand-drawn bins is an important step toward this goal: Such functionality gives students increased freedom in their selection of visual representations and adds another kind of representation for which CLP can provide information to the teacher about students' problem-solving processes. This UAP project has made a valuable contribution in the design, implementation, and testing of a working prototype for machine interpretation of hand-drawn bins—a prototype that now can be easily improved, merged into CLP's code base, and used in future classrooms.

# References

- [1] . Fosnot, C. and Dolk, M. (2001). *Young mathematicians at work: Constructing multiplication and division*. Portsmouth: Heinemann.
- [2] Koile, K. & Rubin, A. (2015) Machine interpretation of students' hand-drawn mathematical representations. In *The Impact of Pen and Touch Technology in Education*. Hammond, T., Valentine, S., Adler, A., and Payton, M. (Eds.) New York: Springer. 49-56.
- [2] Koile, K. and Rubin, A. (2015a) Animated mathematical proofs in elementary education. In *The Impact of Pen and Touch Technology in Education*. Hammond, T., Valentine, S., Adler, A., and Payton, M. (Eds.). Springer, 2015. 67-79.
- [3] Koile, K. and Rubin, A. (2016) Tablet-based technology to support students' understanding of division. To appear in next edition of *The Impact of Pen and Touch Technology in Education*. Hammond, T., Valentine, S., Adler, A., and Payton, M. (Eds). Springer, 2015.
- [4] Rubin, A., Storeygard, J, and Koile, K. (2015) Supporting special needs students in drawing mathematical representations. In *The Impact of Pen and Touch Technology in Education*. Hammond, T., Valentine, S., Adler, A, and Payton, M. (Eds.). Springer, 2015. 57-66
- [5] Yang, D. (2015) Identifying Bins-Dealing Strategies in Classroom Learning Partner. Bachelor of Science Thesis, EECS Department, MIT, Cambridge.

# Appendix A: Student Hand-drawn Bins

Student work with bins drawn as discrete shapes.

**Multiple Choice** (5 x 2 points = 10 points)

Fill in the circle next to the correct answer. Show how you find the answer.

1.  $42 \div 6 = 2 + \underline{\hspace{1cm}}$

(A) 4      (B) 5      (C) 7      (D) 9

**Short Answer** (5 x 2 points = 10 points)

Write the answer in the space given. Show how you find the answer.

6. What is the missing number?

$4 \times 9 = 6 \times \square$

$36$

$36 \div 6 = 6$

**Short Answer**

Write the answer in the space given. Show how you find the answer.

8. What is the missing number?

$\square + 7 = 40 \div 8$

**Short Answer**

Write the answer in the space given. Show how you find the answer.

8. What is the missing number?

$\square + 7 = 40 \div 8$

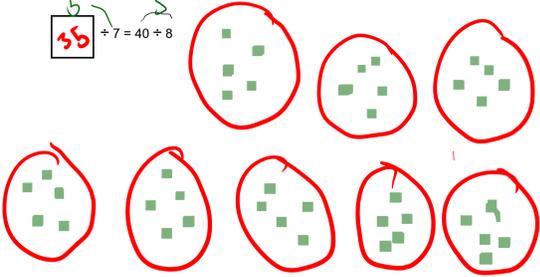
$5$

Short Answer

Write the answer in the space given. Show how you find the answer.

8. What is the missing number?

$\boxed{35} \div 7 = 40 \div 8$



$\boxed{35} \div 7 = 5$   
 $5 \times 7 = 35$

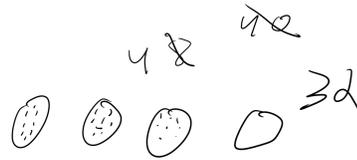


Short Answer

Write the answer in the space given. Show how you find the answer.

10. Amelia makes 48 paper cranes for her teachers. She gives each teacher a string of 8 paper cranes. How many teachers does she give paper cranes to?

\_\_\_\_\_ teachers



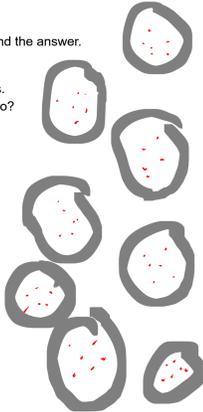
Short Answer

Write the answer in the space given. Show how you find the answer.

10. Amelia makes 48 paper cranes for her teachers. She gives each teacher a string of 8 paper cranes. How many teachers does she give paper cranes to?

6 teachers

$48 \div 8 = 6$

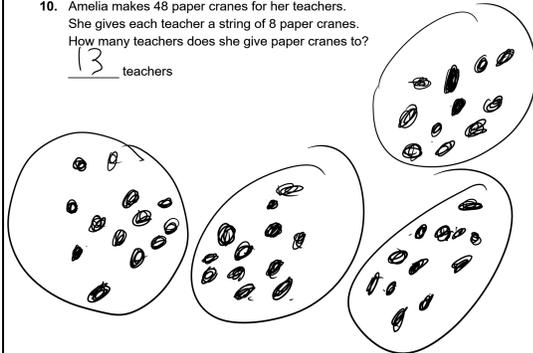


Short Answer

Write the answer in the space given. Show how you find the answer.

10. Amelia makes 48 paper cranes for her teachers. She gives each teacher a string of 8 paper cranes. How many teachers does she give paper cranes to?

13 teachers



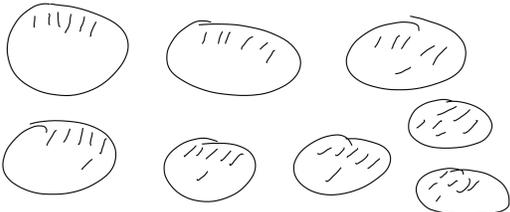
Short Answer

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10. Amelia makes 48 paper cranes for her teachers. She gives each teacher a string of 8 paper cranes. How many teachers does she give paper cranes to?

6 teachers

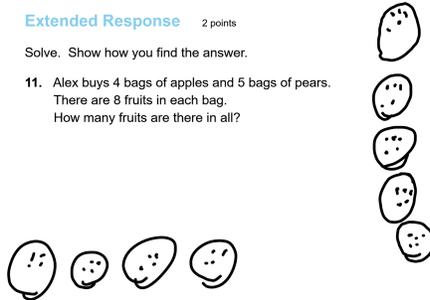
$48 \div 8 = 6$



Extended Response 2 points

Solve. Show how you find the answer.

11. Alex buys 4 bags of apples and 5 bags of pears. There are 8 fruits in each bag. How many fruits are there in all?



There are 47 fruits in all.

Student work with bins drawn as a grid.

**Multiple Choice**

Fill in the circle next to the correct answer. Show how you find the answer.

5. Daniel saves the same amount of money every day.  
He saves \$56 in a week.  
How much money does he save each day?

(A) \$5      (B) \$7      (C) \$8      (D) \$55

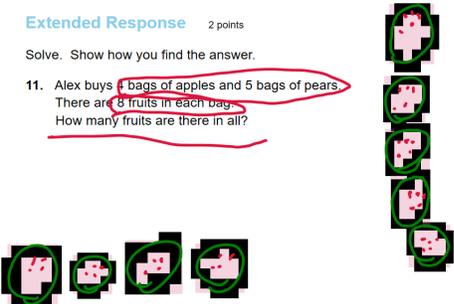
Sun	Mon	Tues	Wednes	Thurs	Fri	Sat
8	8	8	8	8	8	8

# Appendix B: Third Grade Student Results

**Extended Response** 2 points

Solve. Show how you find the answer.

11. Alex buys 4 bags of apples and 5 bags of pears. There are 8 fruits in each bag. How many fruits are there in all?



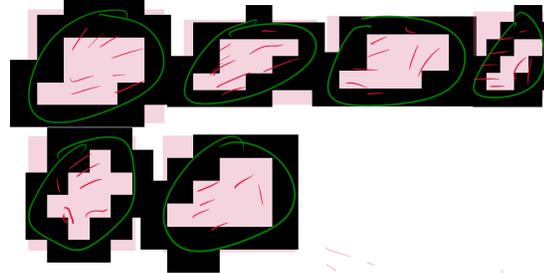
There are 47 fruits in all.

**Multiple Choice** (5 x 2 points = 10 points)

Fill in the circle next to the correct answer. Show how you find the answer.

1.  $42 \div 6 = 2 + \underline{\quad}$

(A) 4 (B) 5 (C) 7 (D) 9

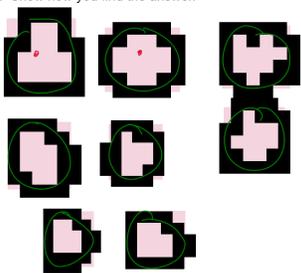


**Short Answer**

Write the answer in the space given. Show how you find the answer.

8. What is the missing number?

$\div 7 = 40 \div 8$

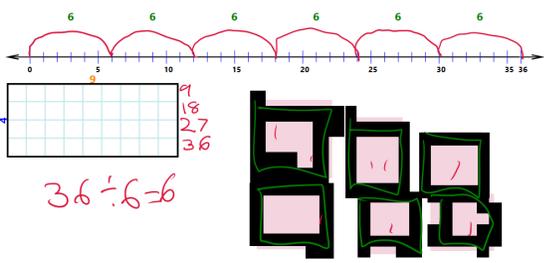


**Short Answer** (5 x 2 points = 10 points)

Write the answer in the space given. Show how you find the answer.

6. What is the missing number?

36  
 $4 \times 9 = 6 \times \square$

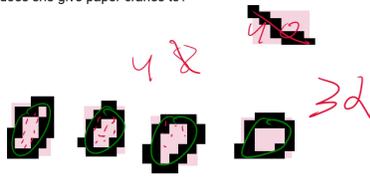


Short Answer

Write the answer in the space given. Show how you find the answer.

10. Amelia makes 48 paper cranes for her teachers. She gives each teacher a string of 8 paper cranes. How many teachers does she give paper cranes to?

\_\_\_\_\_ teachers

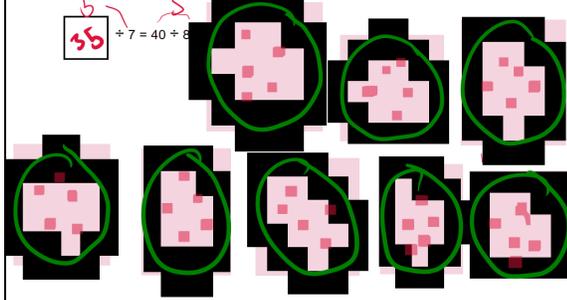


Short Answer

Write the answer in the space given. Show how you find the answer.

8. What is the missing number?

$\boxed{56} \div 7 = 40 \div 8$



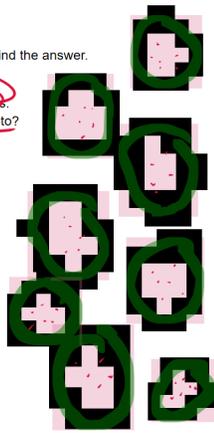
Short Answer

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6 teachers

$48 \div 8 = 6$

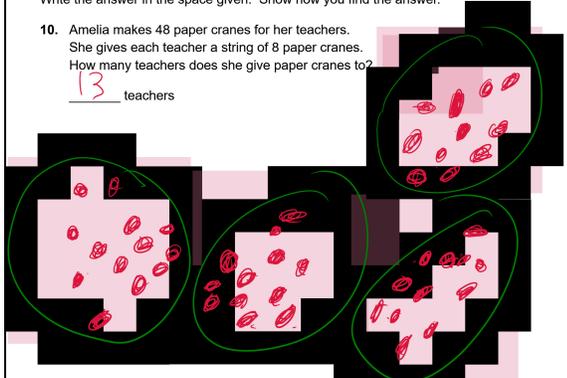


Short Answer

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10. Amelia makes 48 paper cranes for her teachers. She gives each teacher a string of 8 paper cranes. How many teachers does she give paper cranes to?

13 teachers



# Appendix C: Adult Tester Results

10 adult volunteers each worked these five problems:

$24 \div 4$

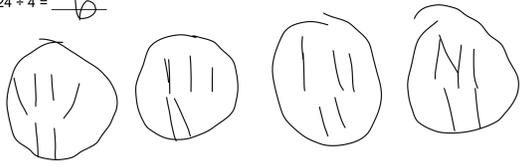
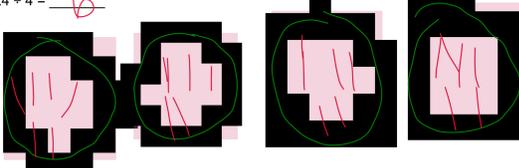
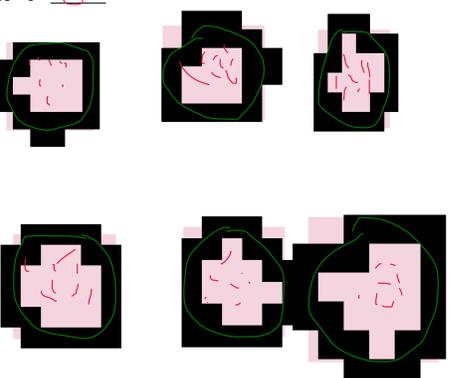
$48 \div 6$

$52 \div 4$

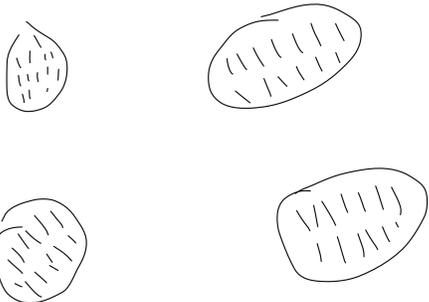
$30 \div 6$

$60 \div 5$

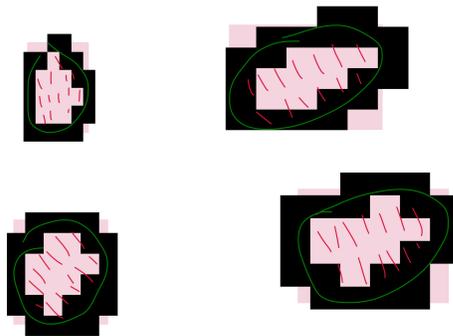
Some examples

<p><math>24 \div 4 = \underline{6}</math></p> 	<p><math>24 \div 4 = \underline{6}</math></p> 
<p><math>48 \div 6 = \underline{8}</math></p> 	<p><math>48 \div 6 = \underline{8}</math></p> 

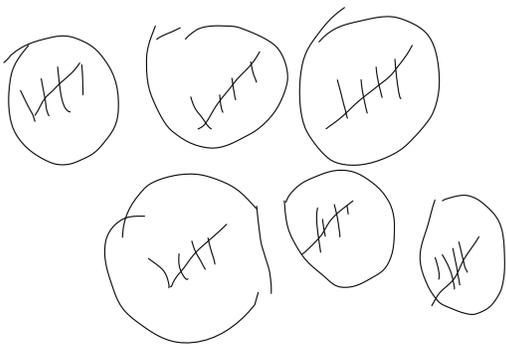
52 + 4 = 13



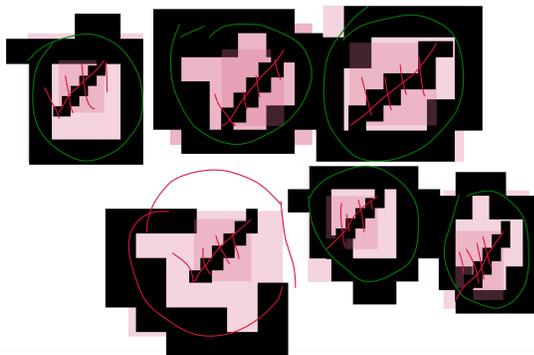
52 + 4 = 13



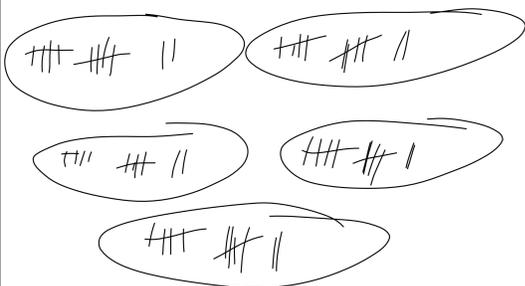
30 + 6 = 5



30 + 6 = 5



60 + 5 = 12



60 + 5 = 12



$52 + 4 = 13$

$52 + 4 = 13$

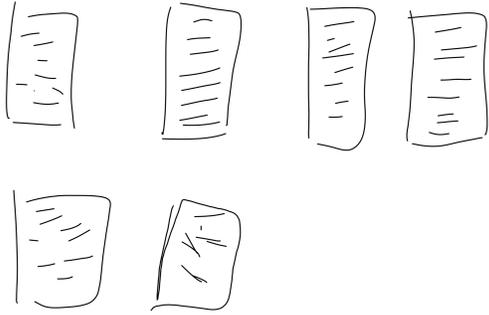
$60 + 5 = 12$

$60 + 5 = 12$

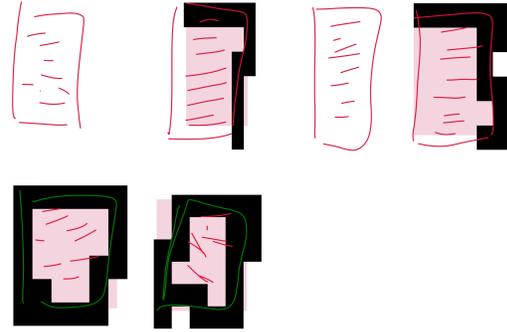
$48 + 6 = 8$

$48 + 6 = 8$

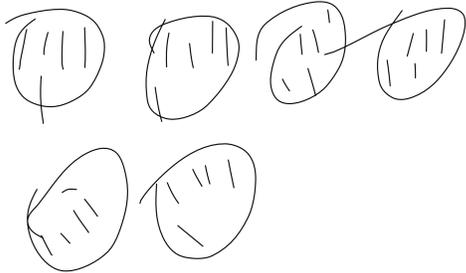
$48 + 6 = \underline{54}$



$48 + 6 = \underline{54}$



$30 + 6 = \underline{36}$



$30 + 6 = \underline{36}$

